Proof of Reachability, Part I.

$X_0 = 0$

$X_1 = [R] \sqcup Pre_1(X_0)$

$X_2 = [R] \sqcup Pre_1(X_1)$

$\vdots$

$X^* = \lim_{n \to \infty} X_n.$

Define

ξ_0 arbitrary.

ξ_1 from $Pre_1(X_0)$

ξ_2 from $Pre_1(X_1)$

ξ_i is the move distr. that yields $Pre_1(X_{i-1})$.

Thm

For all $\varepsilon > 0$, starting from s, I can reach R with prob $\geq X^*(s) - \varepsilon$.

Note: If at s, I play $\xi_i(s)$, the adversary plays η (a distribution of moves),

$$E_s^{\xi_i \eta}(X_{i-1}) \geq Pre_1(X_{i-1}).$$

Here Pl. 2 may play suboptimally.

$$\mathrm{Pre}_1(x_{i-1}) = \sup_{\xi} \inf_{\eta} E^{\xi\eta}(x_{i-1})$$

where

$$E_s^{\xi\eta}(x_{i-1}) = \sum_a \sum_b \sum_t p(s,a,b)(t) \cdot \xi(a) \cdot \eta(b).$$

optimal for pl. 1.

Let $\xi_i^* = \arg \sup_{\xi} \inf_{\eta} E^{\xi\eta}(x_{i-1})$.

Then,
$$E^{\xi_i \eta}(x_{i-1}) \geqslant \inf_{\eta} E^{\xi_i\eta}(x_{i-1})$$

$$= \sup_{\xi} \inf_{\eta} E^{\xi\eta}(x_{i-1}) = \mathrm{Pre}_1(x_{i-1}) \ \underline{\text{ok}}$$

Proof of Thm (Lemma 1 page 10):

You give me $\varepsilon > 0$. I choose $n \in \mathbb{N}$ so that $X_n(s) \geq X^*(s) - \varepsilon$.

Play $\pi_1^{(n)} = \xi_n\, \xi_{n-1}\, \xi_{n-2} \dots \xi_0\, \xi_0\, \xi_0 \dots$

Prove by induction on $n \geq 0$ that:

$$\forall \pi_2 : \quad Pr_s^{\pi_1^{(n)} \pi_2}(\Diamond R) \geq X_n(s).$$

Base case: $X_0 = 0$ obvious.

<u>Ind step</u>: $s \notin R$

prob of $\Diamond R$ from t

$$Pr_s^{\pi_1^{(n)}, \pi_2}(\Diamond R) \geq \sum_t Pr_t^{\pi_1^{(n-1)}, \pi_2[t]}(\Diamond R) \cdot Pr_s^{\pi_1^{(n)} \pi_2}(\text{go to } t \text{ in 1 step}).$$

prob of $s \to t$.

by ind hyp, $\geq$

$$\geq \sum_t X_{n-1}(t) \cdot Pr_s^{\pi_1^{(n)}, \pi_2}(\text{go to } t \text{ in 1 step})$$

$$\geq \sum_t x_{n-1}(t) \cdot \Pr_s^{\xi_n \eta}\left(\begin{array}{c}\text{go to } t\\ \text{in 1 step}\end{array}\right) \quad \text{for some } \eta, \ \eta = \pi_2(s).$$

$$\geq E_s^{\xi_n \eta}(x_{n-1}).$$

$$\geq \mathrm{Pre}_1(x_{n-1})(s) = x_n(s).$$

$$X^* = \mu X. \left([R] \sqcup \mathrm{Pre}_1(X) \right)$$

$$\langle 1 \rangle \Diamond R \geq \mu X. \left([R] \sqcup \mathrm{Pre}_1(X) \right).$$

↑ consequence of the theorem.

Safety Games

Goal: $\Box B$.

$X_0 = \mathbb{1}$

$X_1 = [B] \sqcap Pre_1(X_0) = B$ Prob of B for 1 ~~step~~ moment.

$X_2 = [B] \sqcap Pre_1(X_1)$ max prob of B for 2 moments.

$X_3 = [B] \sqcap Pre_1(X_2)$ max prob of B for 3 moments.

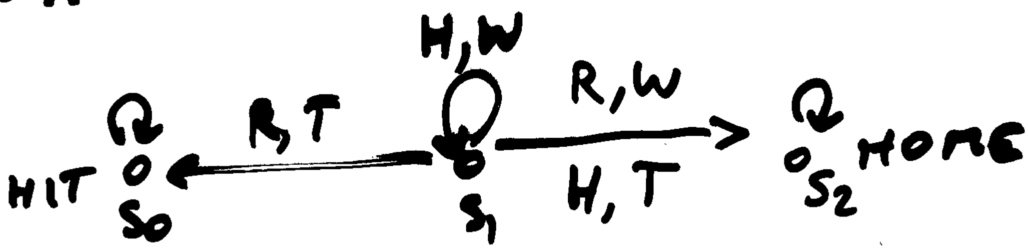
$\vdots$

$$X^* = \lim_{n \to \infty} X_n = \nu X. ([B] \sqcap Pre_1(X)).$$

$$\boxed{\begin{array}{l} \langle 1 \rangle \Box B \overset{=}{\geq} \nu X. ([B] \sqcap Pre_1(X)). \\ \langle 1 \rangle \Diamond B \overset{=}{\geq} \mu X. ([B] \sqcup Pre_1(X)). \end{array}}$$

Homework (next Monday):

Consider HIDE-OR-RUN, and compute



$\Gamma_1(s_1) = \{R, H\}$

$\Gamma_2(s_1) = \{T, W\}$

1) Compute $X_n(s_1)$ for $n \geq 0$.

2) Compute $\xi_n(s_1)$ for $n \geq 0$.