Clarke-Groves-Vickrey

CGV / VCG.

n-players.

- $\mathbb{R}^n$: utilities. $\theta = (\theta_1 - \theta_n)$ (private)

  (θ_i characterizes pl. i
  "type" of i)

- C = set of outcomes = $F \times \mathbb{R}^n$

  F: set of decisions/outcomes
  $\mathbb{R}^n$: payments
  (s, p) s: outc p_i: pay to i.

- There are function v_i, so that the utility u_i to pl. i is: (v_i: public)

$$u_i(s, p) = v_i(s, \theta_i) + p_i$$

- Choice function: $f(\theta) = (s(\theta), p(\theta))$
  where $s(\theta)$ maximizes

$$\sum_i v_i(s(\theta), \theta_i)$$

Theorem

A choice function $f(\theta) = (S(\theta), P(\theta))$ for a CGV environment can be (truthfully) implemented if (and essentially only if):

$$P_i(\theta) = \sum_{j \neq i} v_j(S(\theta), \theta_j) + h_i(\underbrace{\theta_{-i}}_{\theta_1 \cdots \theta_{i-1}\, \theta_{i+1} \cdots \theta_n})$$

For arbitrary functions h_i.

Proof: θ : truthful for i.
θ' : i is lying.

Telling the truth is weakly dominant:

$$\underbrace{u_i(S(\theta), \theta_i)}_{\text{what I get if I tell the truth}} \geq u_i(S(\theta'), \theta_i)$$

$\theta'_j = \theta_j \quad j \neq i$.

Proof by contradiction:

$$u_i(s(\theta'), \theta_i) > u_i(s(\theta), \theta_i)$$

$$v_i(s(\theta'), \theta_i) + \sum_{j \neq i} v_j(s(\theta'), \theta_j') + \cancel{h_i(\theta_{-i}')}$$

$$> v_i(s(\theta), \theta_i) + \sum_{j \neq i} v_j(s(\theta), \theta_j) + \cancel{h_i(\theta_{-i})}$$

or,

$$\sum_j v_j(s(\theta'), \theta_j') > \sum_j v_j(s(\theta), \theta_j)$$

$$\sum_j v_j(s', \theta_j) > \sum_j v_j(s(\theta), \theta_j)$$

there cannot be a better s' than $s(\theta)$ for

Allocation

$F = \{1 \ldots n\}$

$v_i(s, \theta_i) = \delta_{is}\theta_i$

Vickrey Auction

$$s(\theta) = \arg\max_i \theta_i = i_{max}.$$

$$\left(\text{Maximizes } \sum_j v_j(s(\theta), \theta_j)\right).$$

$$P_i(\theta) = \underbrace{\sum_{j \neq i} v_i(s(\theta), \theta_j)} + h_i(\theta_{-i})$$

- if $i = i_{max}$: 0
- if $i \neq i_{max}$: $\theta_{i_{max}}$ $\quad - \max \theta_{-i}$.

$$h_i(\theta_{-i}) = -\max_{j \neq i} \theta_j$$ ~~second price~~

$P_{i_{max}}(\theta) = 0 + (-\text{second price})$ ✓

$P_i(\theta),\ i \neq i_{max} = \theta_{i_{max}} - \theta_{i_{max}} = 0.$

Network Shortest Path.

Players: edges e.

s: decision: a path.

θ_e: cost of edge.

$$v_e(s, \theta_e) = \begin{cases} -\theta_e & \text{if } e \in s \\ 0 & \text{if } e \notin s. \end{cases}$$

Problem:

$$\max \sum_{e \in s} -\theta_e .$$

Payoff:

$$p_e = \begin{cases} 0 & \text{if } e \notin s. \\ \underbrace{\theta_e}_{①} + \underbrace{[dist_{-e} - dist]}_{②} \end{cases}$$

$$h_e(\theta_{-e}) = dist_{-e}$$

① COMPENSATION

② BRIBE/ INCENTIVE TO TELL THE TRUTH.

$$p_e = \sum_{j \neq e} v_j(s, \theta_j) + dist_{-e}$$

Vickrey Auction, revisited.

If i wins:

$$P_i = -\theta_i + (\theta_i - \theta_{second})$$