Lecture 2- Introduction to Zero-sum games.

A two-player game is zero sum if:

$\forall a_1 \in A_1 \quad \forall a_2 \in A_2$:

$$u_1(a_1, a_2) = -u_2(a_1, a_2)$$

Or, since adding a constant to the ~~preferences~~ utility function does not change the preferences:

$$u_1(a_1, a_2) = 1 - u_2(a_1, a_2).$$

<u>Definition</u> (maxminimizers)

$x^* \in A_1$ is a maxminimizer for pl. 1 iff:

$$\min_{y \in A_2} u_1(x^*, y) \geq \min_{y \in A_2} u_1(x, y) \quad \text{for all } x \in A_1.$$

<u>Note</u>: $x^* \in A_1$ is a maxminimizer for pl. 1 iff

$$x^* \in \arg\max_x \min_y u_1(x, y)$$

Similarly, y^* is a maxminimizer for pl. 2 iff:

$$y^* \in \arg\max_y \min_x u_2(x, y).$$

a brief recall ...

For a set $B \subseteq \mathbb{R}$ of reals,

$\sup B = x^*$ such that:

(a). $\forall b \in B, \quad b \leq x^*$

(b). $\forall y \in \mathbb{R}$, if $(\forall b \in B \,.\, b \leq y^*)$,
then $x^* \leq y^*$.

(a) says that x^* is an upper bound for B.

(b) says that x^* is the <u>least</u> upper bound for B.

Fundamental property of the reals:

$$\exists y. \ (\forall b \in B \,.\, b \leq y) \Rightarrow \sup B \ \text{ exists.}$$

"Every bounded set has a least upper bound".

Indeed, $\mathbb{R}$ is defined as the smallest set that includes the rational numbers $\mathbb{Q}$, and that has the above property.

 ... recall continued ...

max B : $x = \max B$ iff:

- $x = \sup B$
- $x \in B$.

max B does not always exist, even if $B \subseteq \mathbb{R}$ and B is bounded. Example:

$$B = \{ x \in \mathbb{R} \mid x^2 < 2 \}$$

or even

$$B = \{ x \in R \mid x < 2 \}.$$

... recall ...

Consider a function $f : A \times B \mapsto \mathbb{R}$
where A and B are arbitrary.

$$\sup_{x \in A} \inf_{y \in B} f(x,y) \quad \overset{>}{\underset{<}{=}} \quad \inf_{y \in B} \sup_{x \in A} f(x,y)$$
??

Think first at: $P(x,y)$: a predicate (true/false) of x, y.
$Q(x)$: of x

$\sup_x g(x)$ is similar to $\exists x . Q(x)$

$\inf_x g(x)$ " " " $\forall x . Q(x)$.

Consider now:

$\exists x \forall y . P(x,y)$ $\quad \leq \quad$ $\forall y \exists x . P(x,y)$

Here, x cannot depend on y

Here, x can depend on y.

"Less true"

"more true".

... recall ...

$$\exists x. \forall y.\ P(x,y) \quad \leqslant \quad \forall y\ \exists x.\ P(x,y)$$

(logical implication) $\Longrightarrow$

$x = g(y)$

x cannot depend on anything: we need a <u>fixed</u> x.

y can choose x looking at y.
g: <u>Skolem</u> function.

"less true" "more true"

The same applies to sup/inf:

$$\sup_{x\in A}\ \inf_{y\in B}\ f(x,y) \leqslant \inf_{y\in B}\ \sup_{x\in A}\ f(x,y)$$

x cannot depend on y.

x <u>can</u> depend on y.

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also, recall:

$$-\max_x f(x) = \min_x\left(-f(x)\right)$$

$$-\sup_x f(x) = \inf_x\left(-f(x)\right)$$

$$\neg\ \forall x\ P(x) = \exists x\ \neg P(x).$$

... back to 0-sum games

Lemma:

Value to pl. 1, 2:

$$v_1 = \max_x \min_y u_1(x,y)$$

$$v_2 = \max_y \min_x u_2(x,y)$$

$$= \max_y \min_x \left(-u_1(x,y)\right)$$

$$= -\min_y \max_x u_1(x,y)$$

We have:

$$\max_x \min_y u_1(x,y) \leq \min_y \max_x u_1(\cancel{x},y)$$

(see recall),

and so:

$$v_1 \leq -v_2$$

or $\quad v_1 + v_2 \leq 0.$

<u>Lemma</u>: For a 0-sum game,

a) If (x^*, y^*) is Nash eq, then x^* is a maxmin, y^* also.

b) If (x^*, y^*) is Nash eq,

$$\max_X \min_Y u_1(x, y) = \min_y \max_X u_1(x, y)$$
$$= u_1(x^*, y^*).$$

So all Nash eq have the same payoff.

c) If $\max_X \min_Y u_1(x, y) = \min_Y \max_X u_1(x, y)$,

$x^* \in \arg \max_X \min_Y u_1(x, y)$

$y^* \in \arg \max_Y \min_X u_2(x, y)$

Then (x^*, y^*) is a Nash eq.

Proof:

(a),(b). Let (x^*, y^*) be a Nash eq.

$$u_2(x^*, y^*) \geq u_2(x^*, y) \qquad \forall y$$

$$u_1(x^*, y^*) \leq u_1(x^*, y) \qquad \forall y \qquad \text{as } u_1 = -u_2$$

Hence,

$$u_1(x^*, y^*) = \min_y u_1(x^*, y) \leq \max_X \min_Y u_1(X, Y).$$

Similarly,

$$u_1(X^*, y^*) \geq u_1(x, y^*) \qquad \forall x$$

$$u_1(x^*, y^*) \geq \min_y u_1(x, y) \qquad \forall x$$

$$u_1(x^*, y^*) \geq \max_X \min_y u_1(x, y).$$

Thus,

$$u_1(X^*, y^*) = \max_X \min_y u_1(x, y)$$

by symmetry

$$u_2(X^*, y^*) = \max_y \min_X u_2(x, y)$$

or also

$$u_1(X^*, y^*) = \min_y \max_X u_1(x, y).$$

This obviously shows also

$$\max_x \min_y u_1(x,y) = \min_y \max_x u_1(x,y)$$

Part (c):

Let $v^* = \max_x \min_y u_1(x,y) = \min_y \max_x u_1(x,y)$.

We have:

$$\max_y \min_x u_2(x,y) = -v^*.$$

$$u_1(x^*,y) \geq v^* \quad \forall y$$
$$u_2(x,y^*) \geq -v^* \quad \forall x.$$

Also,

$$u_1(x^*,y^*) \geq v^*$$
$$u_2(x^*,y^*) \geq -v^* \quad \text{or} \quad u_1(x^*,y^*) \leq v^*$$

So $u_1(x^*,y^*) = v^*$.

This shows that (x^*,y^*) is a Nash Eq.