Existence of Nash Equilibria

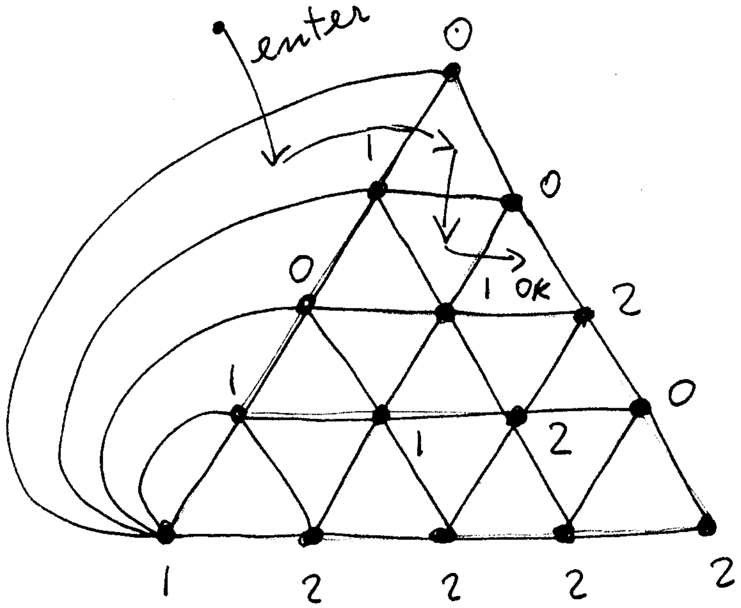
Sperner's Lemma

Consider a subdivided triangle, all whose vertices (and sub-vertices) are labeled with one of 0, 1, 2. Assume also that:

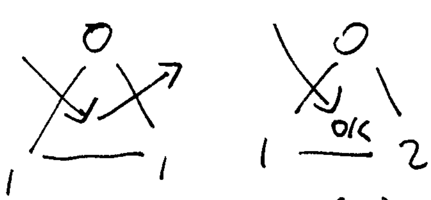
- along the 01 edge, no 2 labels
- along the 12 edge, no 0 labels
- along the 02 edge, no 1 labels.

Then, there is at least one 012-subtriangle.

Proof of Sperner's Lemma:



Close one big edge, then enter, and take a walk as follows:



(a) (b)

(a) If you enter from an ij-edge, and there is another ij-edge, go out of it.

(b) If you found an 012 triangle, be happy!

Notice that:

- There is only one 01-edge, so the walk will not leave.
- Once in a triangle, you can either stay or leave, but if you leave, you will never re-enter: every ~~edge has~~ triangle has at most <u>two</u> edges labeled 01.
- So, the walk must stop in a 012-triangle.

Brower's Fixpoint Theorem

> Let S be an n-dimensional simplex, and
>
> $\varphi : S \mapsto S$ continuous.
>
> Then, $\exists x \in S.\ \varphi(x) = x.$

Simplex: what you can get by stretching (but not cutting) an infinitely elastic triangle.

n-dimensional triangle: convex combination of $n+1$ points in n dimensions:

$$\alpha_0 p_0 + \dots + \alpha_n p_n \qquad \sum_{i=0}^{n} \alpha_i = 1$$

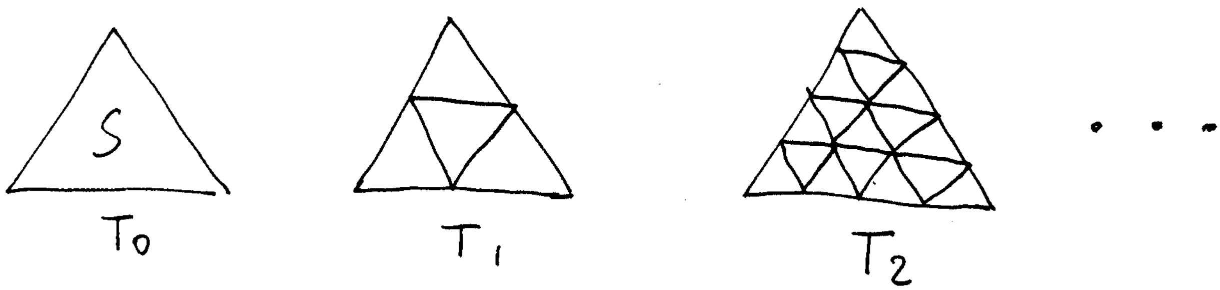
$p_0, \dots, p_n$: points in n dimensions.

Examples of simplex in 2-dim:

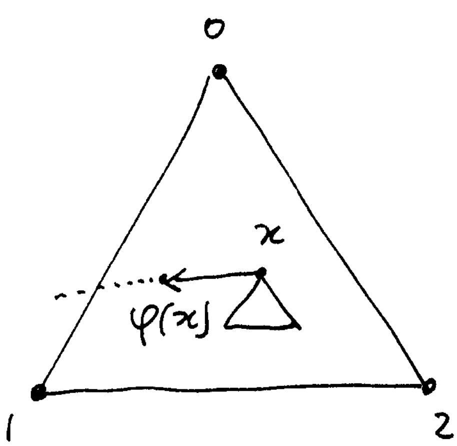
Yes   Yes   Yes   No (has a hole)

Proof of Brower's Theorem (for 2-dim)

Triangulate S into finer and finer subdivisions:

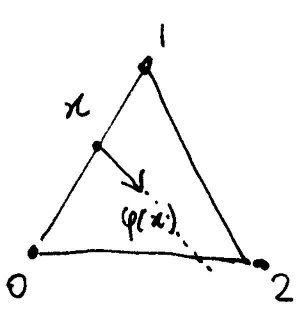


Label each small triangle in T_i as follows:



- x is a vertex of a small triangle.
- Consider the line from x to $\varphi(x)$.
- The line (prolonged) hits the 01 edge of the big triangle.
- So, label x by 2.

Notice that along the 01 external edge, all vertices have a line that hits 02 or 12, and so they will be labeled 0 or 1.



Thus, the labeling is a Sperner labeling.

So, every T_i has a 012-triangle, let x_i be its center (if T_i has more than one 012-triangle, choose one of them at will).

Consider the sequence $x_0, x_1, x_2, \ldots$

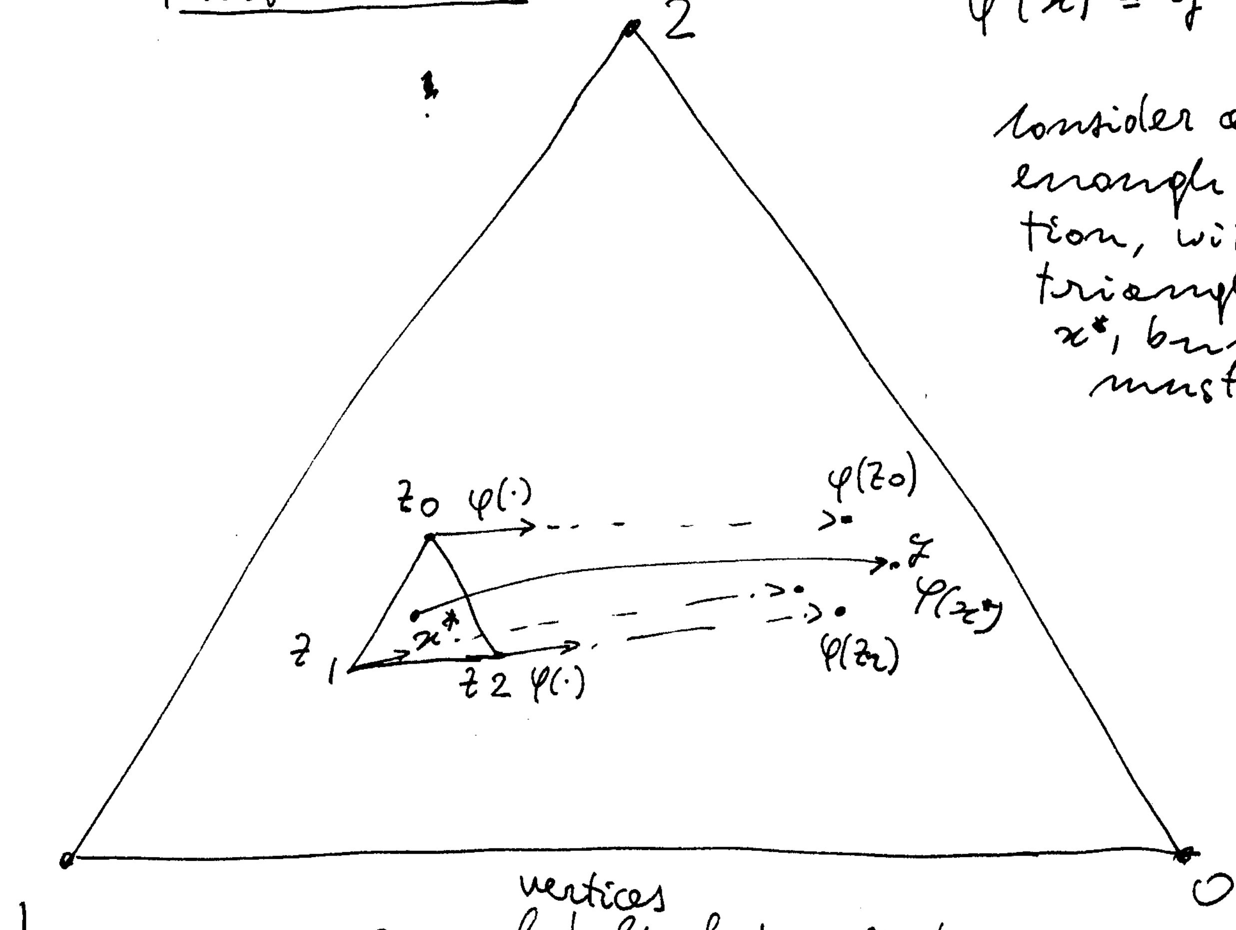
This is a bounded sequence, so by Cauchy's theorem, it has a converging subsequence:

$$x_0', x_1', x_2', \ldots \longrightarrow x^*.$$

<u>Claim</u>: $\varphi(x^*) = x^*$.

<u>Proof of claim</u>: Assume (towards the contradiction) $\varphi(x^*) = y \neq x^*$

Consider a small enough triangulation, with a small triangle around x^*, but not y (y must be far).



z_0 z_1 z_2 : ~~labels~~ vertices of small triangle

z_i ~~has~~ has label i.

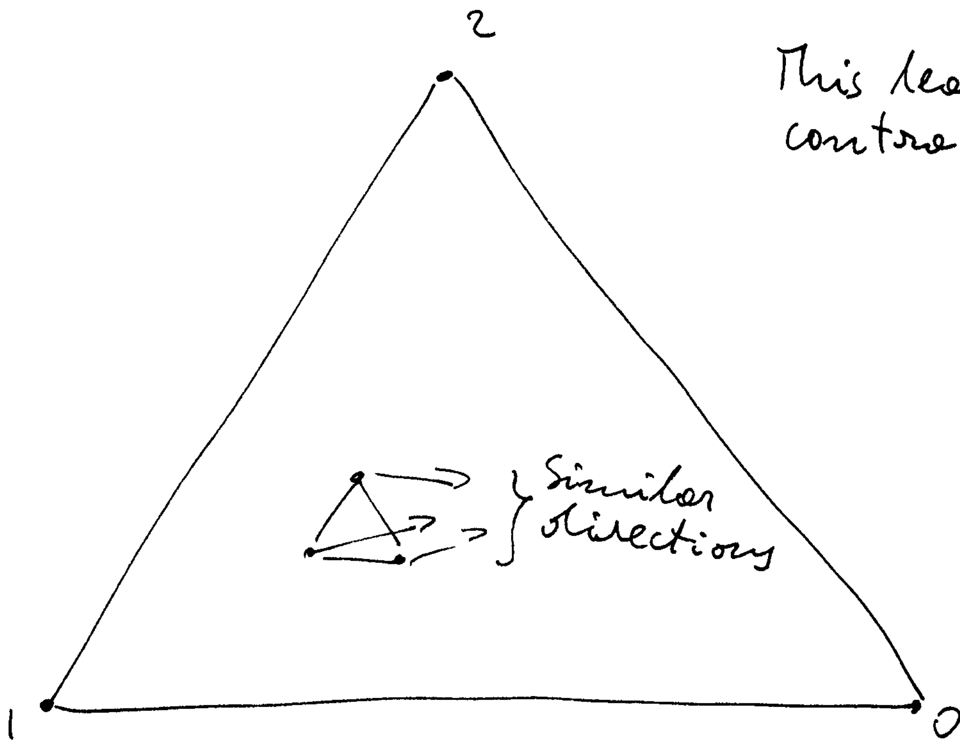
Since φ is continuous, we can choose a small enough triangulation, so that the vertices of the small triangle are mapped very close to y.

But then, it is not possible that the vertices of the small triangle have three different labels, as the prolongations of the

$$z_i \longrightarrow \varphi(z_i)$$

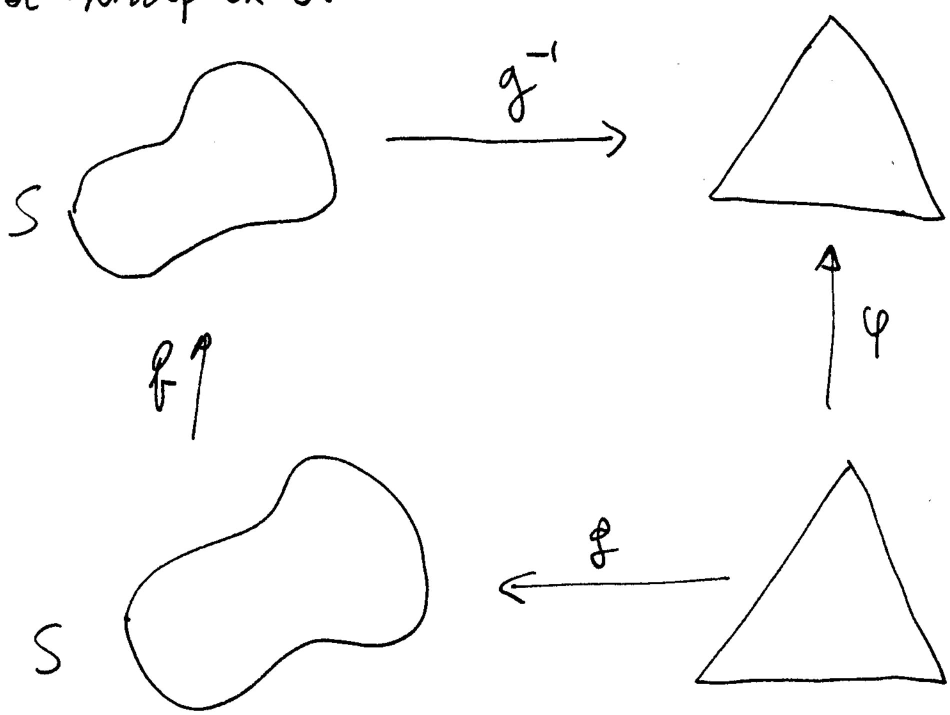
lines can hit at most <u>two</u> of the outer edges:

This leads to the contradiction.



To hit 3 external edges you need two directions to differ by at least $60°$.

Does Brower's Theorem apply only to triangles?
No! Here is how to generalize it (in 2-dim).
Assume we have a continuous $f: S \mapsto S$ for a simplex S.



Since S is a simplex, there is a stretching g that is one-to-one and continuous from a triangle T to S.

Let $\varphi = g; f; g^{-1}$.

$\varphi: T \to T$ is continuous, so ~~it~~ there is $x^* \in T$ with $\varphi(x^*) = x^*$.

<u>Problem</u>:

Show that there is y^* with $f(y^*) = y^*$.

Problem:

- Give a closed bounded set $S \subseteq \mathbb{R}^2$ and a function $\varphi : S \mapsto S$ that is continuous, and such that $\forall x \in S$, $\varphi(x) \neq x$.

- A set $S \subseteq \mathbb{R}^2$ is connected if, for every $x, y \in S$, you can go from x to y without leaving S.
  Give a closed, bounded S, connected, and a continuous $\varphi : S \mapsto S$, such that $\forall x : \varphi(x) \neq x$.

- $\mathbb{R}^2$, obviously, is not bounded.
  Give $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that
  $\forall x \in \mathbb{R}^2.\ f(x) \neq x$.

Kakutani's Fixpoint Theorem

S: Simplex.

$\Phi : S \mapsto 2^S$ such that:

1) $\forall x \in S, \quad \Phi(x) \subseteq S$ is convex.

2) Φ is graph continuous, that is:

assume you have a sequence:

$$(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n), \ldots \longrightarrow (x^*, y^*)$$

(limit)

Such that, $\forall i, \; y_i \in \Phi(x_i)$.

Then, $y^* \in \Phi(x^*)$.

(This is the usual generalization of continuity for ~~graph~~ set-valued functions).

Under these assumptions,

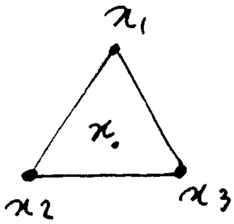
$$\boxed{\exists x^* \in S. \quad x^* \in \Phi(x^*).}$$

Proof of Kakutani's theorem:

Divide S into smaller and smaller triangulations $T_0, T_1, T_2, \ldots$.

Define for each T_i a φ_i as follows:

- For a vertex x of T_i, pick any $\varphi_i(x) \in \Phi(x)$.
- For a non-vertex x, interpolate



Choose $\varphi_i(x)$ as interpolation of $\varphi_i(x_1), \varphi_i(x_2), \varphi_i(x_3)$.

φ_i is continuous, so by Brower's theorem it has a fixpoint: $\exists x_i^* . \; \varphi_i(x_i^*) = x_i^*$.

Consider a converging subsequence

$$\hat{x}_i^*, \;\cdots\; \longrightarrow \hat{x}^*.$$

Claim:
$$\hat{x}^* \in \Phi(\hat{x}^*).$$

<u>Proof of claim:</u>

Recall $\hat{x}^*_i \longrightarrow \hat{x}^*$.

Pick a sequence of triangulation vertices corresponding to $\hat{x}^*_i$, call them $\hat{y}_i$.

$\hat{y}_i \longrightarrow \hat{x}^*$.

Let $\hat{z}_i$ be the value we choose for T_i at $\hat{y}_i$.

$$\hat{z}_i \in \Phi(\hat{y}_i).$$

The sequence of $\hat{z}_i$ may have several accumulation points; if w_k is one such point, by graph continuity, $w_k \in \Phi(\hat{x}^*)$.

~~$\hat{x}_i$ is in the convex combination of its $\hat{z}$~~

~~The result~~

Since $\hat{x}^*_i = \varphi_i(\hat{x}^*_i)$, we have that the limit $\hat{x}^*$ of $\hat{x}^*_i$ must also be in the convex combination of the w_k.

So:

- $w_k \in \Phi(\hat{x}^*)$
- $\hat{x}^*$ is in the convex combination of the w_k.
- $\Phi(\hat{x}^*)$ is convex.

Hence, $\hat{x}^* \in \Phi(\hat{x}^*)$.