Mixed Moves

Penny Matching: a 2-player, 0-sum game with:

u_1

	0	1
0	+1	-1
1	-1	+1

This game has no Nash Equilibrium.

If we play 0, the ~~other~~ adversary plays 1, and vice-versa.
How can we keep the adversary from "guessing"?
We randomize our move choice!

For player i, a <u>mixed</u> move over A_i is a probability distribution x_i over A_i.

When each player $i \in N = \{1 \dots n\}$ plays the mixed move x_i on A_i, we define the value to pl. $i \in N$ as:

$$u_i(x_1 \dots x_n) = u_i(x) =$$

$$= \sum_{a_1 \in A_1} \cdots \sum_{a_n \in A_n} u_i(a_1 \dots a_n)\, x_1(a_1)\, x_2(a_2) \cdots x_n(a_n)$$

$$= \underbrace{E^{x_1 \dots x_n}}_{\text{The expectation, if the players play according to } x_1 \dots x_n} \underbrace{(u_i)}_{\text{of } u_i.}$$

Note: The use of expectations hides a huge assumption: that of linearity, and of risk-neutrality.

• Mixed-move Nash Equilibria

Consider a game $\langle N, A, u \rangle$.

A <u>mixed move</u> Nash equilibrium of the game is a Nash equilibrium of the game $\langle N, X, u \rangle$, where, for each $i \in N$,

X_i = probability distributions over A_i

$u_i(x_1, \dots x_n)$ is defined as before.

The fundamental result is that mixed move Nash equilibria are guaranteed to exist.

<u>Theorem</u>: every game has at least one mixed-move Nash equilibrium.

<u>Proof</u>: Let us do the case for 2 players, as the general case is similar.

Given a (mixed) move profile x, define the best response to x as:

$$BR(x) = (BR_1(x_2), BR_2(x_1)) \quad \text{where } x = (x_1, x_2).$$

$BR : X \mapsto 2^X$ is a set-valued function.
I will show that it satisfies the assumptions of Kakutani's fixpoint theorem.

1) $BR(x)$ is convex for all $x \in X$.

In fact, consider:

$$y, z \in BR(x).$$

For $\alpha \in [0,1]$, I have to show:

$$\alpha y + (1-\alpha) z \in BR(x).$$

In other words, for player 1 (the case for player 2 is symmetrical), I have to show:

$$\alpha y_1 + (1-\alpha) z_1 \in BR_1(x_2)$$

From $y_1 \in BR_1(x_2)$, $z_1 \in BR_1(x_2)$, I get that

$u_1(y_1, x_2) = u_1(z_1, x_2)$. So, by the linearity of expectations, I have:

$$\cancel{u_1(\alpha y_1 + (1-\alpha) z_1) = u_1(y_1, x}$$

$$u_1(\alpha y_1 + (1-\alpha) z_1, x_2) = u_1(y_1, x_2) = u_1(z_1, x_2).$$

Thus, $\alpha y_1 + (1-\alpha) z_1 \in BR_1(x_2)$.

2) BR is graph-continuous.
This is not hard to show; the proof is left as exercise.

As a consequence, there exists at least one $x^* \in X$ with $x^* \in BR(x^*)$,

and such an x^* is a Nash equilibrium.