Computing the value of 0-sum (mixed move) games

$$v_1 = \max_{X_1 \in Distr(A_1)} \; \min_{X_2 \in Distr(A_2)} \; u_1(X_1, X_2)$$

Once pl. 1 has chosen X_1, player 2 can minimize by choosing a pure move. This can be illustrated by the matching pennies game:

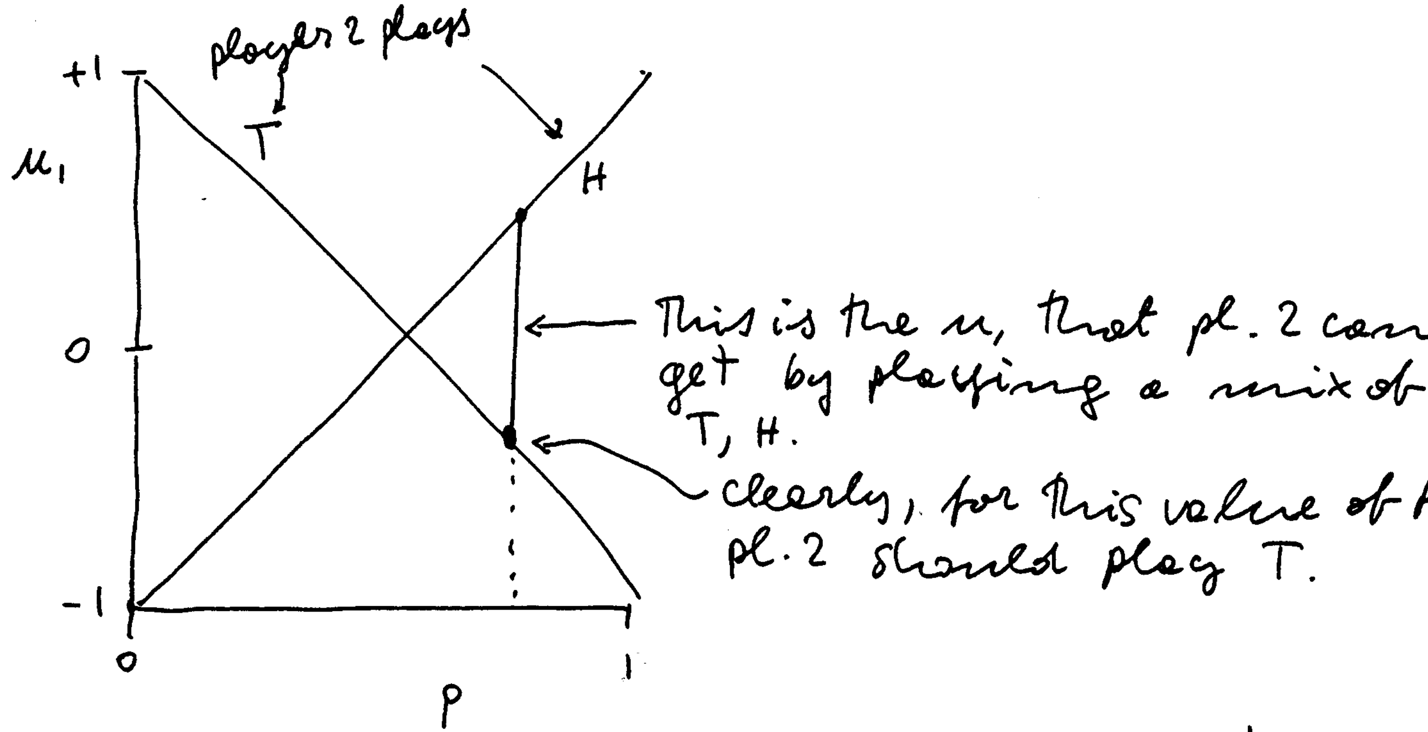
	H	T
H	+1	−1
T	−1	+1

Assume player 1 plays H with prob p, and T with prob $1-p$.

What happens when player 2 selects H, T?

$$u_1(pH + (1-p)T, H) = p - (1-p) = 2p - 1$$

$$u_1(pH + (1-p)T, T) = 1 - 2p$$



The game has a single mixed-move Nash equilibrium, in which both players play with $p = \frac{1}{2}$.

Using this observation, the problem of computing the value of the game for player 1 is reduced to the following:

~~$v_1 = \max_{a_1 \in A_1} \min_{x_2 \in Distr(A_2)} u_1(a_1, x_2)$~~

$$v_1 = \max_{x_1 \in Distr(A_1)} \min_{a_2 \in A_2} u_1(x_1, a_2).$$

Idea: to compute

$$v_1 = \max_{x_1 \in \mathrm{Distr}(A_1)} \; \min_{a_2 \in A_2} \; u_1(x_1, a_2)$$

Consider the set of variables $\{ x_1(a_1) \mid a_1 \in A_1 \}$.

Note: $u_1(x_1, a_2) = \sum_{a_1 \in A_1} x_1(a_1) \cdot u_1(a_1, a_2)$

Solve the following Linear Programming problem to obtain v_1:

$$\boxed{\begin{array}{l} \text{Maximize } y \text{ subject to:} \\[1em] y \leq u_1(x_1, a_2) \qquad\qquad \forall a_2 \in A_2 \\[1em] \displaystyle\sum_{a_1 \in A_1} x_1(a_1) = 1 \\[1em] \forall a_1 \in A_1 : \quad 0 \leq x_1(a_1) \leq 1 \end{array}}$$

ESS = <u>Evolutionary Stable Equilibrium</u>

We assume that all players have the same set of moves, and we reason about the "fraction" of players that play a certain move.

We say that a move b^* is evolutionarily stable iff players do not have an incentive to deviate from it and play another move.

This means that, $\forall b \in A_1$:

$$\underbrace{(1-\varepsilon)\,u(b^*, b) + \varepsilon\, u(b, b)}_{\text{reward to a deviant who plays } b} < \underbrace{(1-\varepsilon)\,u(b^*, b^*) + \varepsilon\, u(b^*, b)}_{\text{reward to the population playing } b^*}$$

So, for all $b \neq b^*$:

- either $u(b, b^*) < u(b^*, b^*)$
- or $u(b, b^*) = u(b^*, b^*)$,

  and $u(b, b) < u(b^*, b)$.

If (b^*, b^*) is a symmetric Nash eq, and no strategy other than b^* is a best response to b^*, then (b^*, b^*) is an ESS.

This game has no ESS:

γ	1	-1
-1	γ	1
1	-1	γ

Mixed strategy Nash eq: play all moves with $\frac{1}{3}$.
Payoff: $\gamma/3$.

A mutant using a pure strategy has payoff

$$\cancel{\tfrac{2}{3}}\,(1-\varepsilon)\,\frac{\gamma}{3} + \varepsilon\gamma$$

So the Nash eq is not evolutionary stable, and the game has no ESS.

Hawks / Doves

	D	H
D	$\frac{1}{2}$ $\frac{1}{2}$	0, 1
H	1, 0	$\frac{1}{2}(1-c)$, $\frac{1}{2}(1-c)$

- If $c > 1$, the game has a unique Nash, with strategy profile $(1 - \frac{1}{c}, \frac{1}{c})$. This is also an ESS.

- If $c < 1$, the unique Nash is to play only H, this is the only ESS.