Reachability and Safety

In a turn-based game with $G = (S, E)$

$S = S_1 \cup S_2$
$E \subseteq S \times S$

Define, for $X \subseteq S$:

$$\begin{aligned} Pre_1(X) = &\{ s \in S_1 \mid \exists t \in S.\ (s,t) \in E \wedge t \in X \} \\ &\cup \\ &\{ s \in S_2 \mid \forall t \in S.\ (s,t) \in E \Rightarrow t \in X \} \end{aligned}$$

Letting $E(s) = \{ t \in S \mid (s,t) \in E \}$

we can write:

$$\begin{aligned} Pre_1(X) = &\{ s \in S_1 \mid \exists t \in E(s).\ t \in X \} \\ &\cup \\ &\{ s \in S_2 \mid \forall t \in E(s).\ t \in X \}. \end{aligned}$$

$Pre_1(X)$: all states that can go to X in one step.

Winning conditions:

$\Phi \subseteq S^\omega$ Φ : a set of infinite sequences of states.

$$\Diamond R = \{ \sigma \in S^\omega \mid \exists k.\ \sigma_k \in R \}$$

(σ_k : state in k-th position in σ, that is, $\sigma = \sigma_0 \sigma_1 \sigma_2 \sigma_3 \cdots$)

set of paths that reach R.

$$\Box R = \{ \sigma \in S^\omega \mid \forall k.\ \sigma_k \in R \}$$

always in R.

$$\Box\Diamond R = \{ \sigma \in S^\omega \mid \sigma_k \in R \text{ for infinitely many } R \}$$

(This is the Büchi condition)

$$\Diamond\Box R = \{ \sigma \in S^\omega \mid \exists k.\ \forall j.\ j \geq k \rightarrow \sigma_j \in R \}$$

(eventually forever in R; this is the coBüchi condition)

Note:

$\Diamond R = S^\omega \setminus \Box \neg R$ where $\neg R = S \setminus R$.

$\Box\Diamond R = S^\omega \setminus \Diamond\Box\neg R$

Strategies:

A strategy for player $i \in \{1,2\}$ is a mapping

$$\pi_i : S^+ \mapsto S$$

such that, $\forall \sigma \in S^*, \forall s \in S, \quad \pi_i(\sigma \cdot s) \in E(s)$.

Outcomes

Given a state s, and two strategies π_1, π_2, an outcome

$s_0, s_1, s_2, \ldots = \text{outcome}(s, \pi_1, \pi_2) \in S^\omega$

is generated according to:

- $s_0 = s$
- $\forall k, \quad s_{k+1} = \begin{cases} \pi_1(s_0 - s_k) & \text{if } s_k \in S_1 \\ \pi_2(s_0 - s_k) & \text{if } s_k \in S_2 \end{cases}$

The successor is decided by the player whose turn it is to play.

Memoryless strategies

π_i is memoryless if, $\forall \sigma \in S^+$, $\forall s \in S$,

$$\pi_i(\sigma s) = \pi_i(s).$$

A memoryless strategy is simply a mapping from states to successor states:

$$\pi_i : S \mapsto S.$$

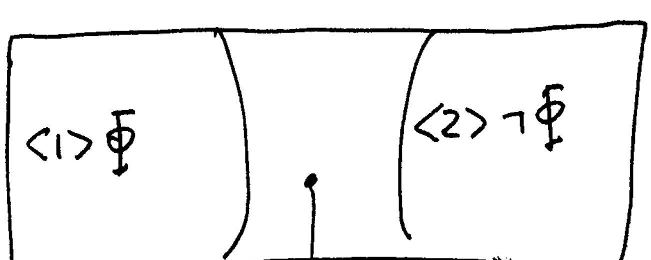
(such that $\forall s \in S$, $\pi_i(s) \in E(s)$ of course).

Winning sets for $\Phi \subseteq S^\omega$:

$$\langle 1 \rangle \Phi = \{ s \mid \exists \pi_1 . \forall \pi_2 . \text{outcome}(s, \pi_1, \pi_2 \in \Phi) \}.$$

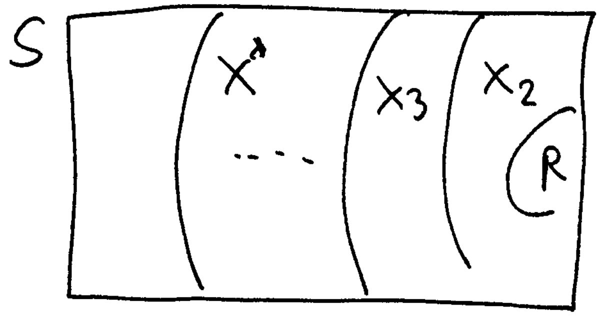
Note that by definition,

$$\langle 1 \rangle \Phi \cap \langle 2 \rangle \neg \Phi = \emptyset.$$



Here, no player can be sure of winning. We will show that this region is empty for many choices of Φ.

Reachability Games



How to compute $\langle 1 \rangle \Diamond R$?

By induction:

$X_0 = \emptyset$

$X_1 = R \cup Pre_1(X_0) = R$ — reaches in 0 steps.

$X_2 = R \cup Pre_1(X_1)$ — reaches in at most 1 step.

$\vdots$

$X_k = R \cup Pre_1(X_{k-1})$ — reaches in at most $k-1$ steps.

$\vdots$

$X^* = \lim_{k \to \infty} X_k$, computable in at most $|S|$ steps.

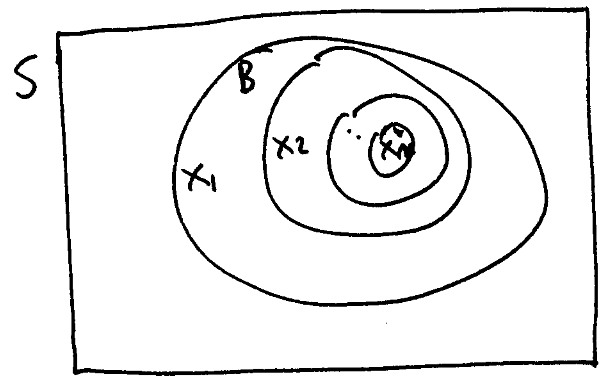
$X^* = \mu X. (R \cup Pre_1(X))$

From the induction,

$$X^* \subseteq \langle 1 \rangle \Diamond R. \qquad (1)$$

Strategy: at $s \in X_k \setminus X_{k-1}$, play according to $Pre_1(X_{k-1})$.

Safety Games



How to compute $\langle 1 \rangle \Box B$?

By induction:

$X_0 = S$

$X_1 = B \cap Pre_1(X_0)$ Can Ys in B

$X_2 = B \cap Pre_1(X_1)$ Can stay in B for at least 1 step.

$\vdots$

$X_k = B \cap Pre_1(X_{k-1})$ Can stay in B for at least 2 steps.

$\vdots$

$X^* = \lim_{k \to \infty} X_k .$

Note: $X^* = B \cap Pre_1(X^*)$

So from X^*, we are in B, and we can be in X^* after one step.

$$X^* \subset \langle 1 \rangle \Box B . \qquad (2)$$

We have:

$$X^* = \nu X . (B \cap Pre_1(X)) \qquad (3)$$

How to show that equality holds in (1) and (2)?
Via μ-calculus complementation:

$$\neg \mu X.\,(R \cup CPre_1(X)) = \nu X.\,(\neg R \cap \neg CPre_1(\neg X)).$$

What is $\neg CPre_1(\neg X)$?

$$\begin{aligned}
\neg CPre_1(\neg X) &= \{s \in S_1 \mid \forall t \in E(s).\ t \notin \neg X\} \\
&\quad \cup \\
&\quad \{s \in S_2 \mid \exists t \in E(s).\ t \notin \neg X\} \\
&= \{s \in S_1 \mid \forall t \in E(s).\ t \in X\} \\
&\quad \cup \\
&\quad \{s \in S_2 \mid \exists t \in E(s).\ t \in X\} \\
&= CPre_2(X)
\end{aligned}$$

(here, Pre = CPre, sorry).

So,

$$\neg \mu X.\,(R \cup Pre_1(X)) = \nu X.\,(\neg R \cap Pre_2(X))$$

So, from (1):

$$X^* = \mu X.\,(R \cup Pre_1(X)) \subseteq \langle 1 \rangle \Diamond R.$$

$$\neg X^* = \nu X.\,(R \cap Pre_2(X)) \underbrace{\subseteq}_{\text{by comparison with (2).}} \langle 2 \rangle \Box R.$$

<u>Hence:</u>

$$\boxed{\begin{aligned} \langle 1 \rangle \Diamond R &= \mu X.\,(R \cup Pre_1(X)) \\ \langle 1 \rangle \Box R &= \nu X.\,(R \cap Pre_1(X)) \end{aligned}}$$

Büchi and CoBüchi games

$\langle 1 \rangle \Box \Diamond B$?

Let us reach B once:

$$\langle 1 \rangle \Diamond B = \mu X. \begin{pmatrix} \neg B \cap Pre_1(X) \\ \cup \\ B \end{pmatrix}$$

Once we reach, we want to be able to do it again.
Assume we are lucky enough to find Y
such that:

$$Y = \mu X. \begin{pmatrix} \neg B \cap Pre_1(X) \\ \cup \\ B \cap Pre_1(Y) \end{pmatrix}$$

Then, clearly $Y \subseteq \langle 1 \rangle \Box \Diamond B$.
Which Y should we use? The largest!
So,

$$\nu Y. \mu X. \begin{pmatrix} \neg B \cap Pre_1(X) \\ \cup \\ B \cap Pre_1(Y) \end{pmatrix} \subseteq \langle 1 \rangle \Box \Diamond B.$$

How do we prove equality?
Via μ-calculus complementation.

$$\neg \nu Y. \mu X. \begin{pmatrix} \neg B \cap Pre_1(X) \\ \cup \\ B \cap Pre_1(Y) \end{pmatrix}$$

$$= \mu Y. \nu X. \begin{pmatrix} \neg B \cap Pre_2(X) \\ \cup \\ B \cap Pre_2(Y) \end{pmatrix}$$

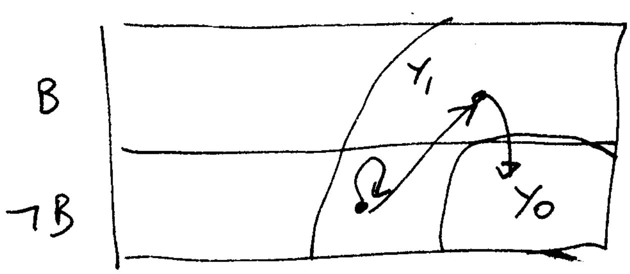
What does the latter formula mean?
Let us compute:

$Y_0 = S$ (ok)

$$Y_1 = \nu X. \begin{pmatrix} \neg B \cap Pre_2(X) \\ \cup \\ \underbrace{\cancel{B \cap Pre_2(Y_0)}}_{\text{empty}} \end{pmatrix} = \langle 2 \rangle \Box \neg B$$

by comparison with the safety formula.

$$Y_2 = \nu X. \begin{pmatrix} \neg B \cap Pre_2(X) \\ \cup \\ B \cap Pre_2(Y_1) \end{pmatrix}$$

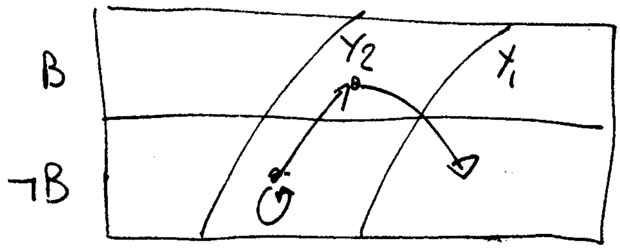


While in Y_2:

- If in $\neg B$, stay in Y_2.
- If in B, go to Y_0.

$Y_2 = \langle 2 \rangle$ "at most one visit to B".

$$Y_3 = \nu X. \begin{pmatrix} \neg B \cap Pre_2(X) \\ \cup \\ B \cap Pre_2(Y) \end{pmatrix}$$



while in Y_2:

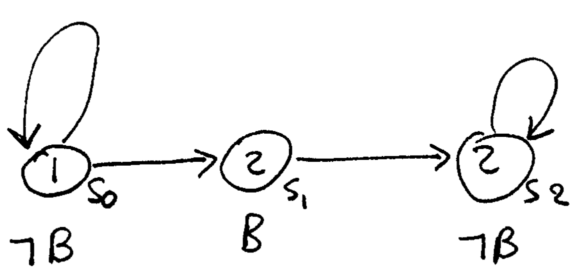
- If in $\neg B$, stay in Y_2
- If in B, go to Y_1.

$Y_3 = \langle 2 \rangle$ "at most two visits to B".

So, in the limit,

$$\mu Y. \nu X. \begin{pmatrix} \neg B \cap Pre_2(X) \\ \cup \\ B \cap Pre_2(Y) \end{pmatrix} = \langle 2 \rangle \Diamond \Box \neg B.$$

<u>note</u>: Player 2 may be able to win, but player 2 may <u>not</u> be able to control when the final $\neg B$ begins:



Player 2 wins from all states, but cannot control whether to win staying at s_0, or going to s_2, and the "indecision" may last forever if pl. 2 wishes.