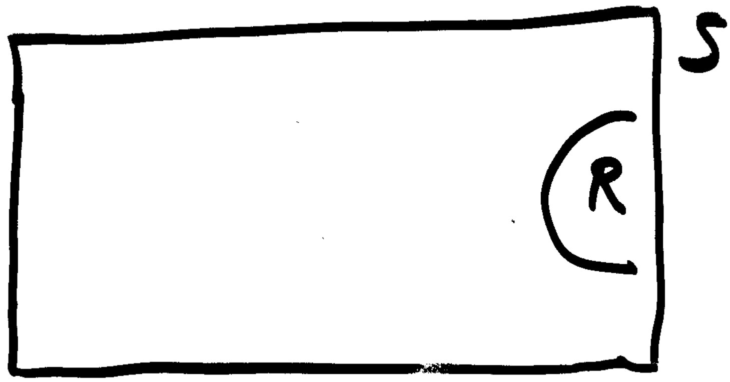
Concurrent Reachability

$G = \langle S, M, \Gamma_1, \Gamma_2, p \rangle$

$R \subseteq S$



$$\langle 1 \rangle \Diamond R = \sup_{\pi_1} \inf_{\pi_2} \Pr^{\pi_1, \pi_2}(\Diamond R)$$

<u>Valuation</u> : $v : S \mapsto \mathbb{R}$ (or $S \mapsto [0,1]$)

Pointwise operations :

$V = S \mapsto \mathbb{R}$ (all valuations).

$v_1, v_2 \in V$

$v_1 + v_2$ as

$$(v_1 + v_2)(s) = v_1(s) + v_2(s)$$

$v_1 \underset{\sqcap}{\sqcup} v_2$ as $(v_1 \underset{\sqcap}{\sqcup} v_2)(s) = v_1(s) \underset{\sqcap}{\sqcup} v_2(s)$

$v_1 \leq v_2$ iff $\forall s: v_1 \leq v_2$

$v_1 < v_2$ iff $v_1 \leq v_2$ and $\exists s \in S.\ v_1(s) < v_2(s)$.

We want to shift thinking from ~~value~~ subsets of states to valuations.

For $T \subseteq S$, $[T] : S \mapsto \mathbb{R}$

$$[T](s) = \begin{cases} 1 & \text{if } s \in T \\ 0 & \text{othw.} \end{cases}$$

LEAST

$0 = \lambda s.0$ = the function f st. $f(s) = 0$ $\forall s \in S$.

$1 = \lambda s.1$

GREATEST valuation.

$(V, \leq)$ is a complete lattice.

Set-based reachability.

$X_0 = \emptyset$

$X_1 = R$

$X_2 = R \cup Pre_1(R)$

$X_3 = R \cup Pre_1(X_2)$

$\vdots$

Now, with probabilities:

$X_0 = 0$

$X_1 = [R]$

$X_2 = [R] \sqcup PPre_1([R])$

$X_3 = [R] \sqcup PPre_1(X_2)$

$PPre_1(X)$ is the max. expectation of X I can guarantee in 1 step.

In a turn-based game,

$$[\,Pre_1(X)\,] \quad = \quad PPre_1(\,[X]\,)$$

For a turn-based game,

$$[\mu X.\,(Pre_1(X) \cup R)\,]$$
$$= \mu X.\,(\,PPre_1(X) \sqcup [R]\,)$$

Thus, $\mu X.\,(\,[R] \sqcup PPre_1(X)\,)$ $\qquad$ (1)

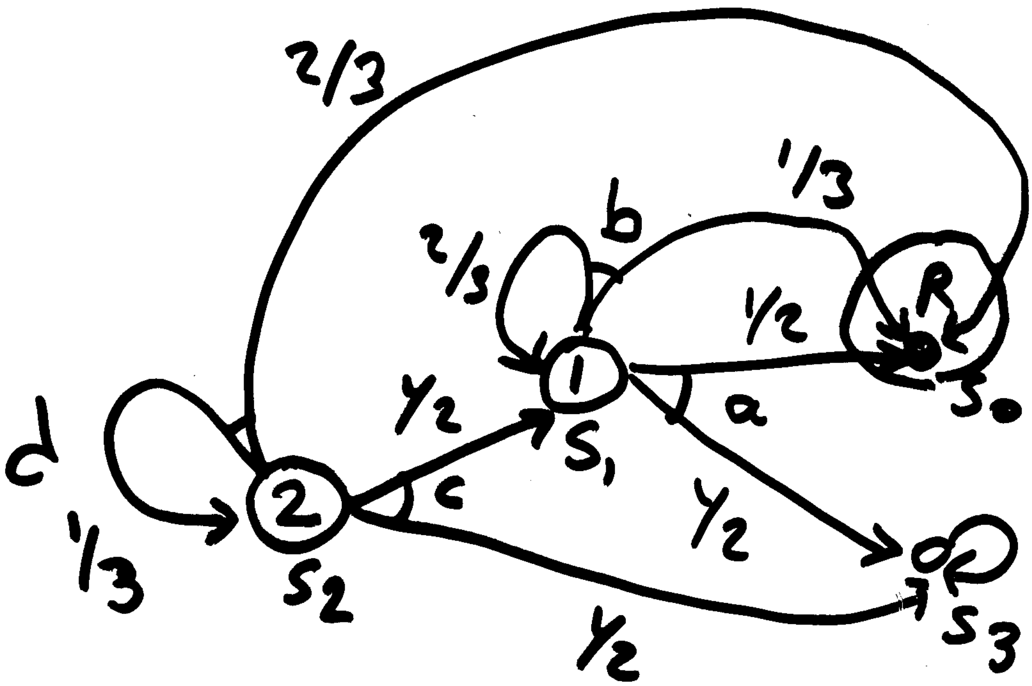
gives the max prob. of winning a reachab. game at least for turn-based games.

Can we show (1) works also for concurrent games?

Yes. See later.

Example

· on a stochastic turn-based game.



x_n	s_1	s_2	s_0
x_0	0	0	0
$x_1 = [R]$	0	0	1

$x_2 = \mathrm{PPre}(x_1) \cup [R]$

$x_2 =$	$1/2$	0	1
	$a \rightsquigarrow 1/2$ ●	$c \rightsquigarrow 0$ ●	
	$b \rightsquigarrow 1/3$	$d \rightsquigarrow 2/3$	

$s_1 \qquad s_2 \qquad s_0$

$X_3 = \mathrm{PPre}_1(X_2) \sqcup [R]$

$X_3 = \quad \frac{2}{3} = v_3(s_1) \qquad \frac{1}{4} = v_3(s_2) \qquad 1$

$a \rightsquigarrow \frac{1}{2}$

$b \rightsquigarrow \frac{1}{3} + \frac{2}{3} \cdot \frac{1}{2} = \frac{2}{3}$

$c \rightsquigarrow \frac{1}{2} \cdot \frac{1}{2}$

$d \rightsquigarrow \frac{2}{3}$

$X_4 = \mathrm{PPre}_1(X_3) \sqcup [R]$

$X_4 = \quad \frac{7}{9} \qquad \frac{1}{3} \qquad 1$

$a \rightarrow \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 0 = \frac{1}{2}$

$c \rightsquigarrow \frac{1}{2} \cdot \frac{2}{3} = \cancel{2}\,\frac{1}{3}$

$\frac{2}{3}$ — $v_3(s_1)$

$b \rightarrow \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot \frac{2}{3} = \frac{7}{9}$

$d = \frac{2}{3} + \frac{1}{3} \cdot \frac{1}{4}$

$\frac{1}{4}$ — $v_3(s_2)$