CMPE 248 Winter 2007

Homework 1, due Thursday January 18, 2007

1. Consider a game $G = \langle S, M, \Gamma_1, \Gamma_2, \delta \rangle$, and a set $R \subseteq S$. We define two sequences of sets $\{X_k\}_{k \geq 0}$ and $\{Y_k\}_{k \geq 0}$ as follows, for $k \geq 0$:

$$X_0 = \emptyset \qquad\qquad Y_0 = R$$
$$X_{k+1} = R \cup \mathrm{Pre}_1(X_k) \qquad\qquad Y_{k+1} = Y_k \cup \mathrm{Pre}_1(Y_k)$$

   Prove that $\lim_{k \to \infty} X_k = \lim_{k \to \infty} Y_k$.

2. Let $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$. Is $(\mathbb{N}_\infty, \leq)$ a complete lattice? Explain. (The ordering is defined so that, for all $x \in \mathbb{N}_\infty$, we have $x \leq \infty$).

3. Given a monotonic function $f : L \mapsto L$ over a lattice $(L, \leq)$, define $X_k = f^k(\bot)$ for $k \geq 0$, where f^k as usual denotes f applied k times. Let $X_* = \sqcup_{k \geq 0} X_k$. Prove that $X_* \leq \mu X.f(X)$.

4. Given the lattice $(2^S, \subseteq)$ of subsets of S, for $R \subseteq S$ indicate with $\neg R$ the set $\neg R = S \setminus R$. Consider a monotonic function $f : 2^S \mapsto 2^S$. Prove that

$$\neg \mu X.f(X) = \nu X.\neg f(\neg X). \tag{1}$$

5. Given a lattice $(L, \leq)$, we say that a function $g : L \mapsto L$ is *monotonically decreasing* if, for all $x, y \in L$, we have that $x \leq y$ implies $f(x) \geq f(y)$. Suppose that you want to prove the following theorem:

   > Consider a lattice $(L, \leq)$ and a monotonic function $f : L \mapsto L$. Let $\neg : L \mapsto L$ be a monotonically decreasing function such that (A), and assume that f is such that (B). Then,
   >
   > $$\neg \mu X.f(X) = \nu X.\neg f(\neg X).$$

   What are the weakeset assertions you can take for (A) and (B) that preserve the spirit of (1)? For (A), what do you need to know about $\neg$? For (B), do you need f to be continuous?