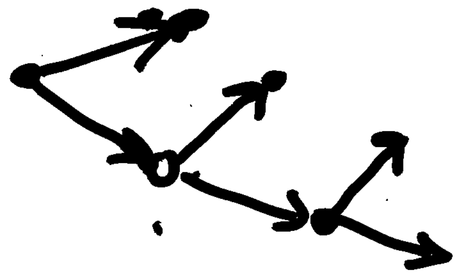
Game Theory

games over graphs

2 players (or 1)

0-sum

Markov Decision Processes



game relations

n-player non-0-sum

- multi-player games
- one-shot
- Nash eq.
- Auctions - mech des.
- Networks
- Reputations

Game Structure

- a finite state space S
- a finite set M of moves.
- Two move assignments:

$$\Gamma_1 : S \mapsto 2^M \setminus \emptyset$$

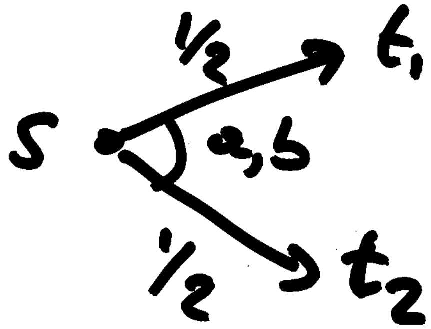
$$\Gamma_2 : S \mapsto 2^M \setminus \emptyset .$$

- a transition function

$$\delta : S \times M \times M \mapsto Distr(S)$$

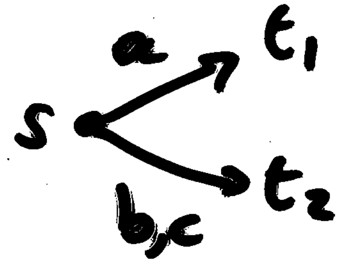
Ex: $\delta(s, \underset{\text{pl.1 move}}{a}, \underset{\text{pl.2 move}}{b}) = \begin{cases} \frac{1}{2} & t_1 \\ \frac{1}{2} & t_2 \end{cases}$

pl.1 move   pl.2 move   prob



Types of Game Structures

- Graphs = 1 player, determ.
  - $\forall s \in S: \ |\Gamma_2(s)| = 1.$
  - $\forall s \in S \ \forall a \in \Gamma_1(s).$
    $\exists t. \ \delta(s, a, -)(t) = 1.$



- Markov Chain

- $\forall s, \ |\Gamma_1(s)| = |\Gamma_2(s)| = 1.$

- MDP: $\forall s \ |\Gamma_2(s)| = 1.$

- <u>Deterministic</u>:

  $\forall s, \forall a, b \in M, \ \exists t$:

  $$\delta(s, a, b)(t) = 1 .$$

- <u>Turn-Based</u>

  There is $\gamma : S \longmapsto \{1, 2\}$

  s.t., $\forall s, \ \left|\Gamma_{\sim\gamma(s)}(s)\right| = 1.$

$\sim 1 = 2$
$\sim 2 = 1$

- DT = D + T.

Strategy:

$\pi_1 : S^+ \mapsto \mathrm{Distr}(M)$

such that:

if $\pi(s_0 \dots s_n)(a) > 0$,

then $a \in \Gamma_1(s_n)$.

Similarly for π_2.

- Deterministic:

  $\forall \sigma \in S^+$, $\pi(\sigma)$ is determ.

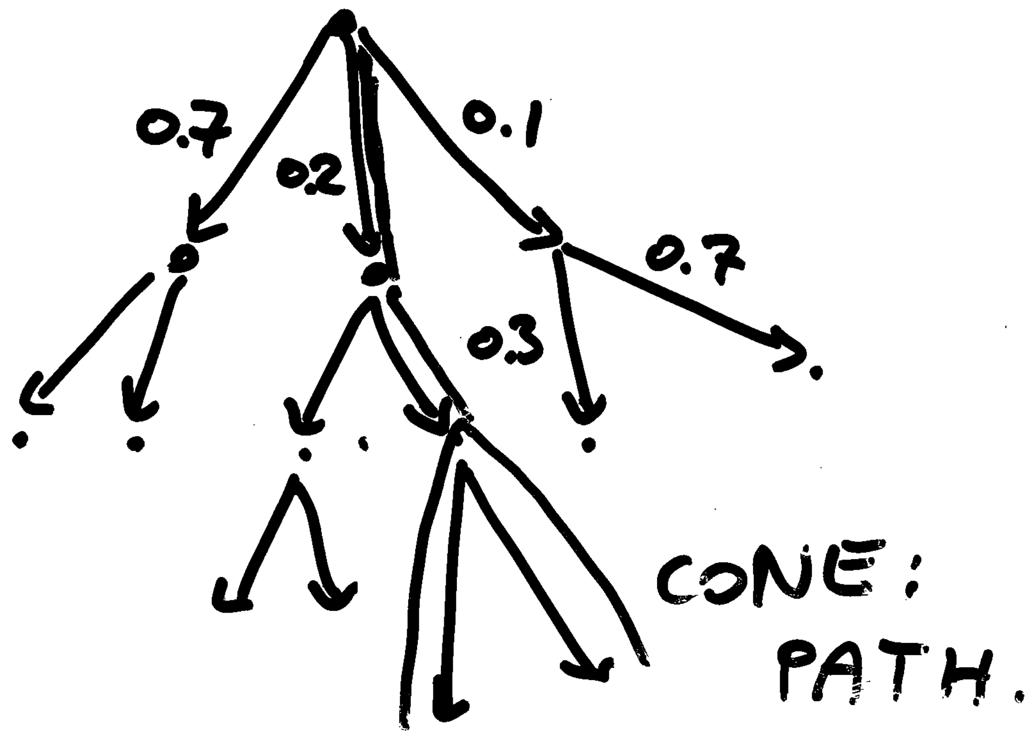
- Memoryless:

  $\forall \sigma \in S^* \quad \forall s \in S$,

  $\pi(s) = \pi(\sigma \cdot s)$

$\Pi \quad \Pi^M \quad \underset{\text{"pure"}}{\Pi^D} \quad \Pi^{MD}$ "simple" "pure"

Stochastic Processes in a Nutshell

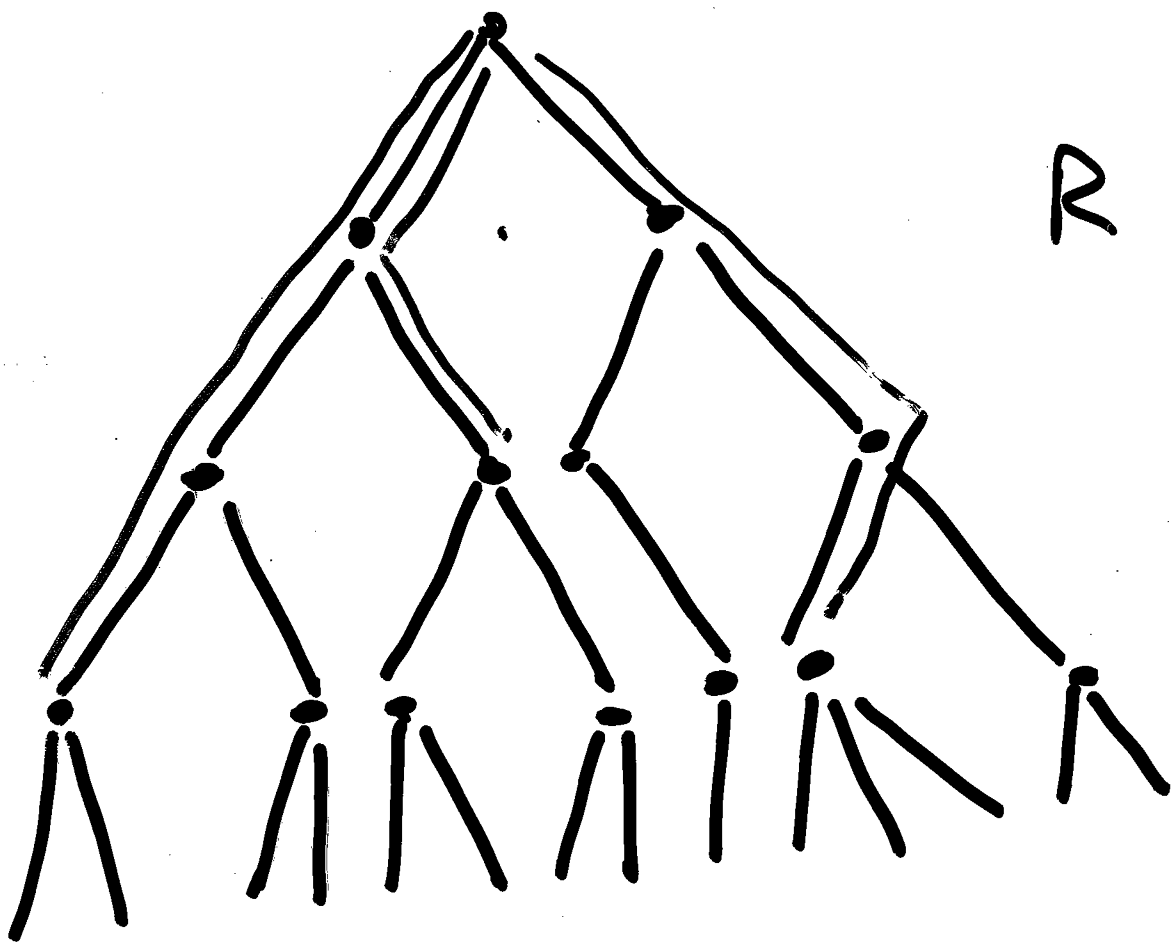


- CONES HAVE a well-def (easy to compute/define) probability; multiply prob. ~~leading~~ along path.

Readability:

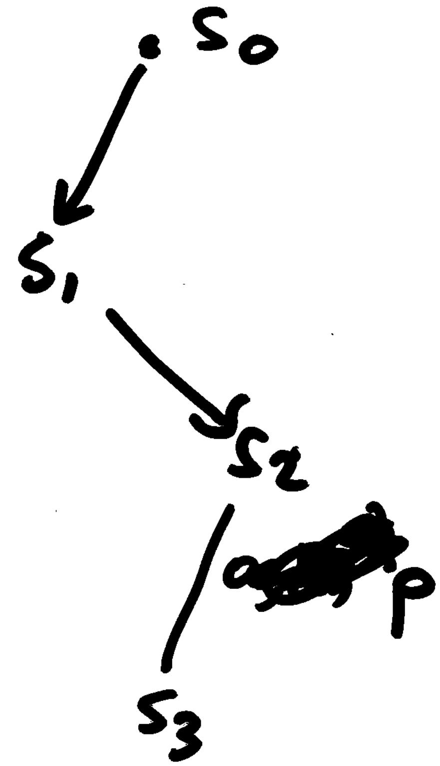
given $R \subseteq S$,

$$\Diamond R = \{ \sigma \in S^\omega \mid \exists k \geq 0.\ \sigma_k \in R \}$$



Events = measurable sets of paths: so start from the set of cones, and close w.r.t. countable union, intersection, set difference.

Given $\mathcal{G}, \pi_1, \pi_2$, we have a stochastic process!



$$p = \sum_{a \in \Gamma_1(s_2)} \sum_{b \in \Gamma_2(s_2)} \pi_1(s_0 .. s_2)(a) \cdot$$

$$\cdot\, \pi_2(s_0 \dots s_2)(b) \cdot \delta(s_2, a, b)(s_3).$$

$$\underset{\text{start}}{Pr_s^{\pi_1, \pi_2}} (\underbrace{\Diamond R}_{\text{event}}) \in [0, 1].$$

Deterministic TB-games

$R \subseteq S$

$\Diamond R = \{ \sigma \in S^\omega \mid \exists k \geq 0\ \sigma_k \in R \}$

$\Box R = \{ \sigma \in S^\omega \mid \forall k \geq 0.\ \sigma_k \in R \}$

$$\sigma \in \Diamond R \ \underline{\text{iff}} \ \sigma \notin \Box (S \setminus R)$$

$\langle 1 \rangle \Diamond R = \{ s \mid \exists \pi_1 . \forall \pi_2 .$
$\quad \text{outcomes}(s, \pi_1, \pi_2) \subseteq \Diamond R \}$

$\langle 1 \rangle \Box R = \{ s \mid \exists \pi_1 . \forall \pi_2 .$
$\quad \underline{\text{outcomes}}(s, \pi_1, \pi_2) \subseteq \Box R \}$

$\langle i \rangle \varphi = \{ s \mid \exists \pi_i . \forall \pi_{\neq i}$
$\quad \text{outcomes}(s, \pi_1, \pi_2) \subseteq \varphi \}$

$\varphi \subseteq S^\omega.$

~~$s_0 s_1$~~

$$s_0 \longrightarrow s_1 \longrightarrow s_2 \xrightarrow{a,b} s_3 \longrightarrow \cdots$$

$\in \text{outcomes}(s_0, \pi_1, \pi_2)$

iff : $\cancel{\exists \pi_1, \pi_2}$

$\forall k,\ \exists a \in \Gamma_1(s_k)$

$\exists b \in \Gamma_2(s_k)$ s.t.:

$\pi_1(s_0 \dots s_k)(a) > 0$

$\pi_2(s_0 \dots s_k)(b) > 0$

$\delta(s_k, a, b)(s_{k+1}) > 0$.