Reachability

$$\mu X.\ ([R] \sqcup QPre_1(X))$$

$\neg$: $f \in V$ $\quad f : S \mapsto [0,1]$.

$\neg f$: $\neg f(s) = 1 - f(s) \quad \forall s \in S$.

$$\neg \mu X.\ ([R] \sqcup QPre_1(X))$$

$$\wr$$

$$\nu X.\ ([\neg R] \sqcap \neg QPre_1(\neg X))$$

Lemma 1 $\quad \neg QPre_1(\neg X) = QPre_2(X)$

so,

$$\nu X.\ ([\neg R] \sqcap QPre_2(X)) \leq \langle\langle 2 \rangle\rangle \Box \neg R.$$

Lemma 2.

Lemma 0:

$$\mu X.\ ([R] \sqcup QPre_1(X)) \leq \langle\langle 1 \rangle\rangle \Diamond R.$$

Lemmas 0 + 1 + 2 =

Theorem:

$$\langle\langle 1 \rangle\rangle \Diamond R = \mu X. ([R] \sqcup \mathrm{QPre}_1(X))$$
$$\langle\langle 2 \rangle\rangle \Box R = \nu X. ([R] \sqcap \mathrm{QPre}_2(X))$$

Proof Let $q = [\mu X. ([R] \sqcup \mathrm{QPre}_1(X))](s)$.

Lemma 0: $(\langle\langle 1 \rangle\rangle \Diamond R)(s) \geq q$.

Lemma 2: $(\langle\langle 2 \rangle\rangle \Box \neg R)(s) \geq 1 - q$
(+1)

Lemma 1 ds.1

$$1 - \acute{Q}Pre_1(1-X) = QPre_2(X)$$

Fix any $s \in S$.

$$(1 - QPre_1(1-X))(s) =$$

$$1 - \sup_{\xi_1 \in D_1(s)} \inf_{\xi_2 \in D_2(s)} \left(1 - E_s^{\xi_1,\xi_2}(X)\right)$$

von Neumann's minimax Thm.

$$= 1 - \inf_{\xi_2} \sup_{\xi_1} \left(1 - E_s^{\xi_1,\xi_2}(X)\right)$$

$$= \cancel{1} + \sup_{\xi_2} \inf_{\xi_1} \left(E_s^{\xi_1,\xi_2}(X) - \cancel{1}\right)$$

$$= \sup_{\xi_2} \inf_{\xi_1} E_s^{\xi_1,\xi_2}(X) = QPre_2(X)(s). \quad \underline{\underline{ok}}$$

Lemma 2

$$\langle\langle 2 \rangle\rangle \Box \neg R \geqslant \nu X . \left([\neg R] \sqcap QPre_2(X) \right)$$

Proof

$$X_0 = 1 \qquad (\text{ds. } 1)$$

$$X_1 = [\neg R] \sqcap QPre_2(1) = [\neg R]$$

$$X_2 = [\neg R] \sqcap QPre_2(X_1)$$

$$X_3 = [\neg R] \sqcap QPre_2(X_2)$$

...

$X_i(s) \leqslant$ prob of staying in $\neg R$ for i steps

....

$$X_*(s) = \nu X . ([\neg R] \sqcap QPre_2(X))(s) \leqslant \langle\langle 2 \rangle\rangle \Box \neg R .$$

ok.

$$\langle\langle 1 \rangle\rangle \Box\Diamond R = \nu Y. \mu X. \left(\begin{array}{c} [\neg R] \sqcap QPre_1(X) \\ \sqcup \\ [R] \sqcap QPre_1(Y) \end{array} \right)$$

<u>Tail winning condition:</u>

"pure future"

A winning condition is <u>tail</u> if, for all winning σ, and for all t in σ:

That is:

$$\sigma : s_0 \xrightarrow{\sigma'} t \xrightarrow{\sigma''} \text{winning.}$$

then also ~~the~~ all sequences

$$\bullet \xrightarrow{\eta} t \xrightarrow{\sigma''}$$

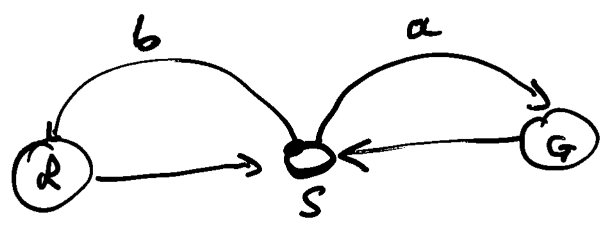
for <u>all</u> η are winning.

What are examples of tail winning conditions?

- <u>not</u> $\Box R$, $\Diamond R$.
- Büchi.
- co Büchi
- Parity.

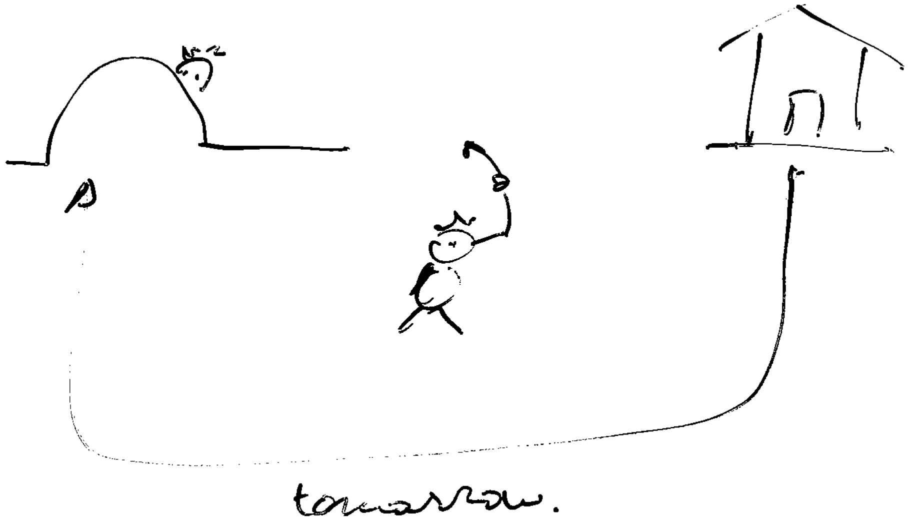
Goal $R, G \subseteq S$

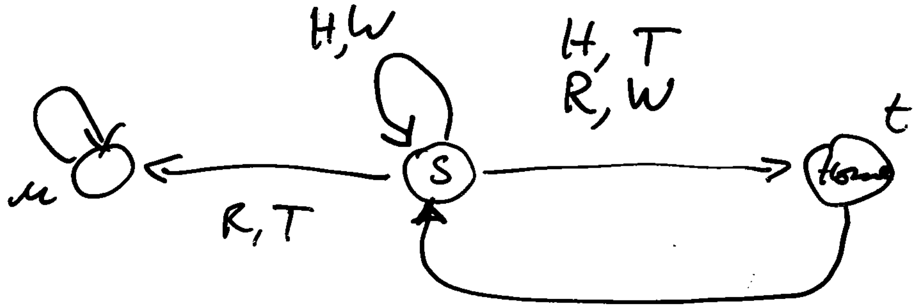
$$\Box\Diamond R \wedge \Box\Diamond G. = \varphi$$



- $\exists \pi_1$ w. finite memory (1 bit), determ, st $\forall \pi_2$. $\forall \sigma \in$ outc(s, π_1, π_2), φ holds.

- $\exists \pi_1$, plays $\frac{1}{2}a + \frac{1}{2}b$, st. $\forall \pi_2$. $Pr_s^{\pi_1 \pi_2}(\varphi) = 1$.





$\varphi : \Box \Diamond \text{Home}.$

Assume π_1 memoryless.

Call p the prob. of running at s.

- Pl. 1 goes home once with prob $\geq 1-p$.
- Pl. 2 hits with prob. $\geq p$.

$$\inf_{\pi_2} \Pr_s^{\pi_1, \pi_2}(\varphi) = 0.$$

Can we win with $R_2 > 0$ using memory?

At every cycle, if we choose $R = p$, we ~~win~~ home $\geq (1-p)$.

~~Wins two~~

Home twice: $(1-p_1) \cdot (1-p_2)$

$$\varphi: \quad \prod_{i=1}^{\infty} (1-p_i)$$

Can we choose the p_i to make the above > 0?

$$\mathrm{QPre}_1(X)(s) = \max_{\xi_1} \min_{\xi_2} \overbrace{\sum_{t \in S} \sum_{a \in \Gamma_1(s)} \sum_{b \in \Gamma_2(s)} X(t) \cdot \xi_1(a) \cdot \xi_2(b) \cdot \tau(s,a,b)(t)}^{E_s^{\xi_1, \xi_2}(X)} = 1\,0\,/$$

$$= \max_{\xi_1} \min_{b \in \Gamma_2(s)} \sum_{t \in S} \sum_{a \in \Gamma_1(s)} X(t) \cdot \xi_1(a) \cdot \tau(s,a,b)(t)$$

In LP: $\{y\} \cup \{\xi_1(a) \mid a \in \Gamma_1(s)\}$

maximize y subject to:

$$\forall b \in \Gamma_2(s): \quad y \leq \sum_{t \in S} \sum_{a \in \Gamma_1(s)} X(t) \cdot \xi_1(a) \cdot \tau(s,a,b)(t)$$

$$\forall a \in \Gamma_1(s): \quad 0 \leq \xi_1(a) \leq 1$$

$$\sum_{a \in \Gamma_1(s)} \xi_1(a) = 1$$

Let π_1^* be s.t. 0-sum game

(1) $\forall \pi_2.\quad v(\pi_1^*, \pi_2) \geq \sup_{\pi_1} \inf_{\pi_2'} v(\pi_1, \pi_2')$

π_2^* st.

(2) $\forall \pi_1 \quad v(\pi_1, \pi_2) \leq \inf_{\pi_2} \sup_{\pi_1} v(\pi_1, \pi_2)$.

Then,

$$\pi_1^* \in BR_1(\pi_2^*)$$

$$\pi_2^* \in BR_2(\pi_1^*).$$

(1) Let $\pi_1^* = \arg\sup_{\pi_1} \inf_{\pi_2} v(\pi_1, \pi_2)$

$$\pi_2^* = \arg\inf_{\pi_2} \sup_{\pi_1} v(\pi_1, \pi_2)$$

Are Reachability games in P?

Can we reduce them to LP?

$$\langle\langle 1\rangle\rangle \Diamond R = \mu X.\ \left([R] \sqcup QPre_1(X)\right)$$

We need the least X : $S \mapsto \mathbb{R}$ st.

$$\begin{cases} X(s) = 1 & \text{if } s \in R. \\ X(s) = QPre_1(X)(s) & \text{if } s \notin R. \end{cases}$$

$$X(s) = \max_{\xi_1} \min_{b \in \Gamma_2(s)} \sum ---$$

We cannot say:

"maximize $X(s)$ subject to

$X(s) \leq \sum ---$

. . ."

The global minimization (2) clashes with the local max (1) — no LP reduction (we know, as values can be irrational).

MDPs

Reachability:

$$\langle 1 \rangle \quad \Diamond R = \mu X \left([R] \sqcup \mathrm{QPre}_1(X) \right)$$

$$\mathrm{QPre}_1(X)(s) = \sup_{\xi_1} \sum_{a} \sum_{t \in S} \tau(s,a)(t)\, \xi_1(a)\, X(t)$$

$$= \max_{a \in \Gamma_1(s)} \sum_{t \in S} \tau(s,a)(t) \cdot X(t)$$

Reachability in MDPs via LP: $\{ X(s) \mid s \in S \}$

minimize $\sum_{s} X(s)$ subject to:

$\forall s \in R \quad X(s) = 1$

$\forall s \notin R, \ \forall a \in \Gamma_1(s) \quad X(s) \geq \sum_{t \in S} \tau(s,a)(t) \cdot X(t)$

So, ~~complexity~~ problem is in P.

$\boxed{\text{Safety}}$

$$\langle\langle 1\rangle\rangle \Box R \;=\; \nu X.\left([R] \sqcap QPre_1(X) \right)$$

Maximize X subj to:

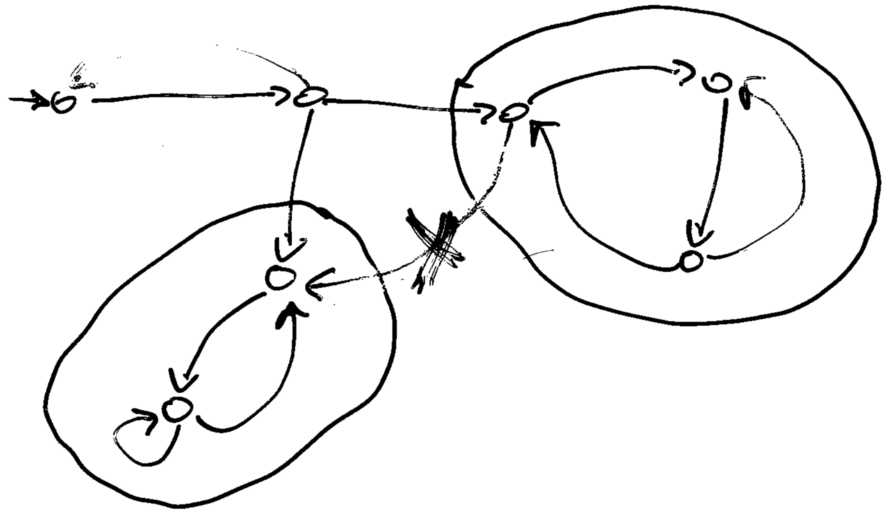
$$X(s) = 0 \qquad s \notin R$$

$s \in R$: $$X(s) = \max_{a \in \Gamma_1(s)} \sum_t \tau(s,a)(t) \cdot X(t).$$

$$X(s) \leq 1.$$

Cannot be done, directly, via LP.

Markov Chain

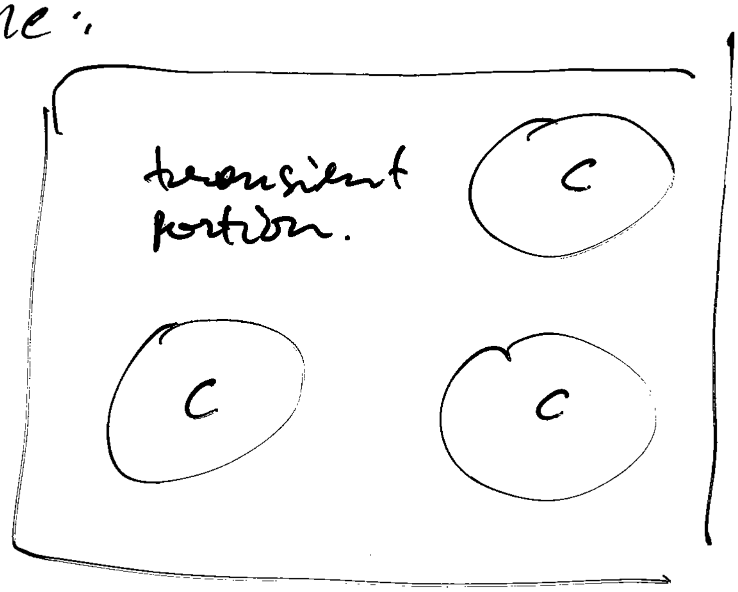


What are the sets C such that

$$\Pr(\mathrm{infi}(\sigma) = C) > 0 \quad ?$$

answer: the closed strongly connected components.

Me:



C: closed str.com