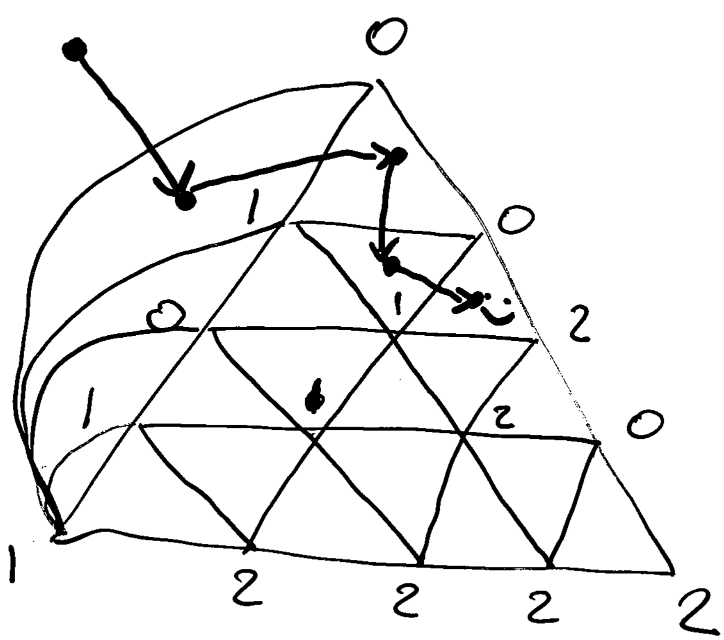
Existence of Nash Equilibria

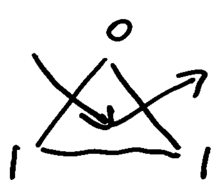
Sperner's Lemma

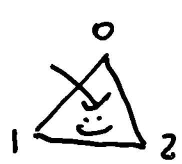
Triangle, subdivided, labeled by 0, 1, 2 st
- Vertices are labeled 0, 1, 2
- Along 01 edge, no 2 labels, etc.

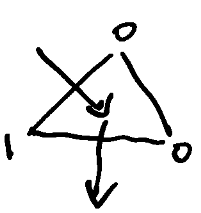
Then, there is at least one 012-subtriangle.



Rules







1) In my walk, I never go out.

2) The degree of any triangle in the walk is at most 2.

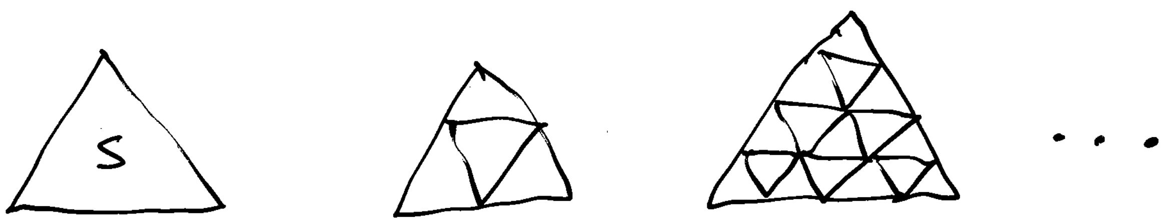
So, by the wool ball theorem, the walk ends in a degree 1, 012-subtriangle.

<u>Browers' Fixpoint theorem.</u>

Let S be an n-dimensional simplex,
and $\varphi: S \longmapsto S$, continuous.
Then, $\exists\, x: \quad x = \varphi(x)$.

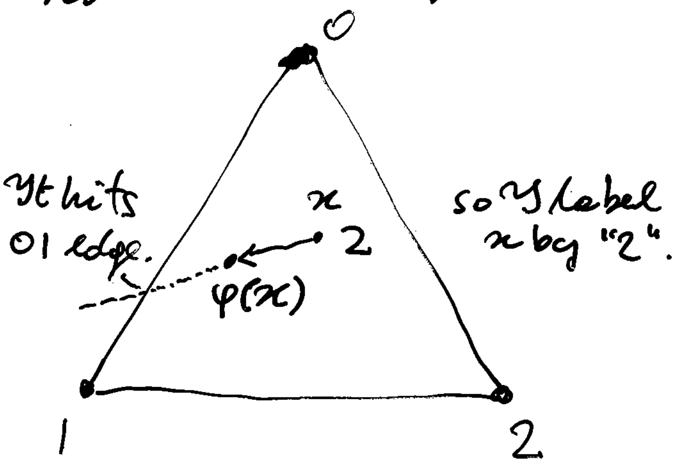
Simplex = triangle, in n dimensions
(~~linear span~~ convex comb. of $n+1$ vectors)
can be stretched.

Proof: ~~The~~ Triangulate S into finer and finer triangulations:

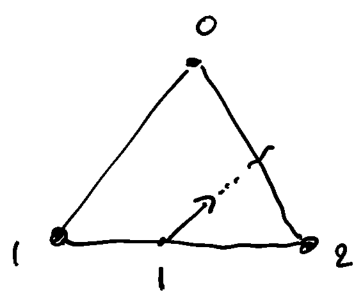


$T_0 =$ $\qquad T_1 \qquad T_2$

Label a triangulation as follows:



Every T_i with these labels becomes a Sperner Triangle.



Every T_i has a 012-subtriangle, let x_i be its center.

Consider the sequence $x_0, x_1, x_2, x_3, \ldots$

We have a converging subsequence

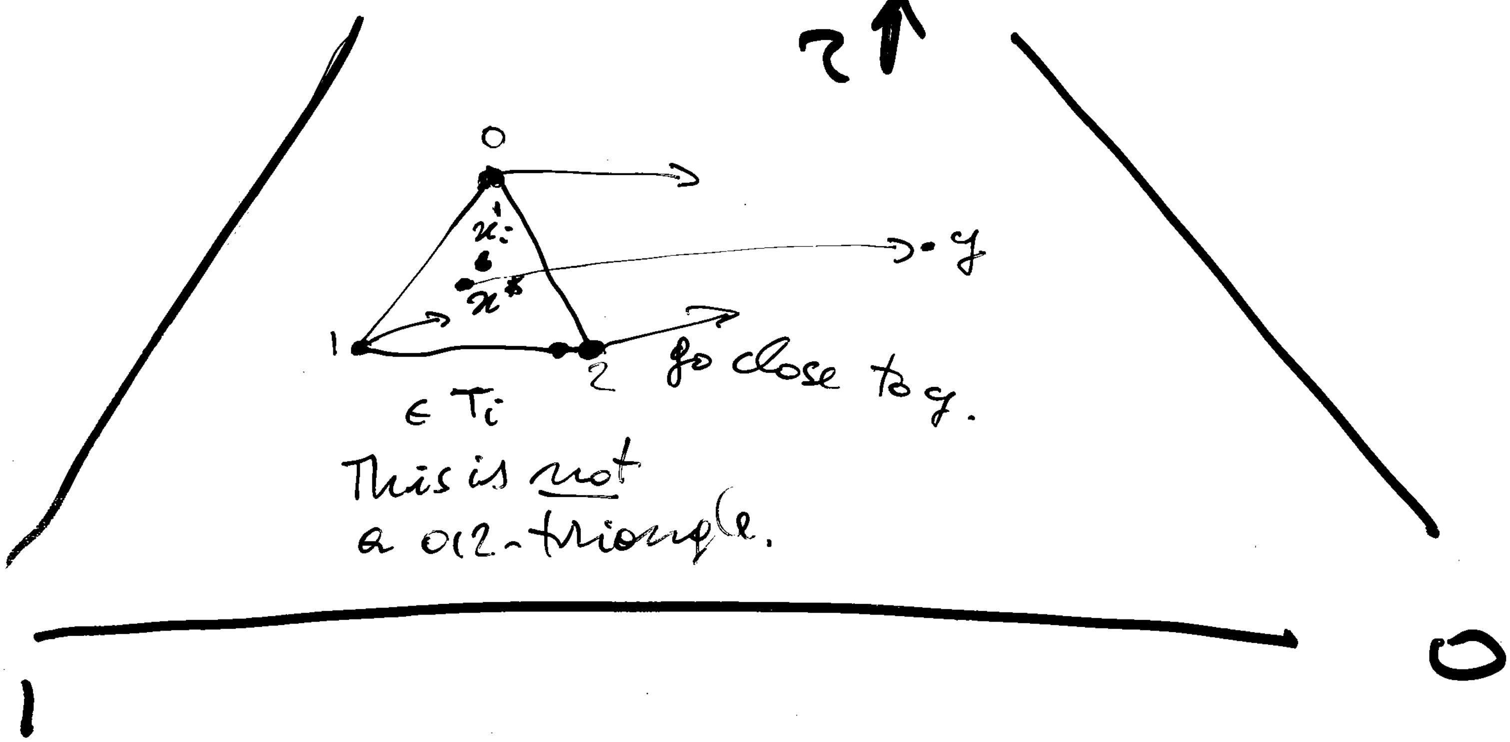
$$x'_0, x'_1, x'_2, \longrightarrow x^*.$$

Claim: $\varphi(x^*) = x^*$.

~ Proof:

Assume $\varphi(x^*) = y \neq x^*$.

Consider a small enough triangulation:



"You cannot have a 012 triangle whose vertices are all mapped in the same far away general region".

Kakutani's Fixed Point Theorem.

S: simplex.

$\Phi : S \mapsto 2^S$

1). convex-valued
2). graph-continuous

1) $\forall s$, $\Phi(s)$ is convex.

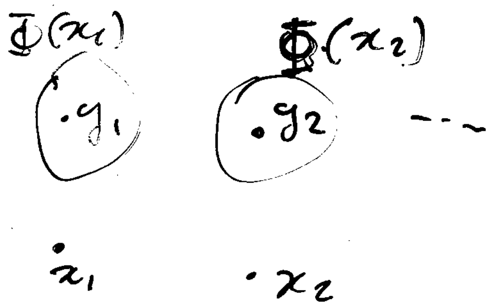
2) Assume you have a sequence

$$(x_0, y_0), (x_1, y_1), (x_2, y_2) \cdots \rightarrow (x, y)$$

Such that, $\forall i$,

$$y_i \in \Phi(x_i).$$

Then, $y \in \Phi(x)$



Then, $\exists x^*$: $x^* \in \Phi(x^*)$.

<u>Proof</u> $T_0, T_1, T_2, \ldots$ smaller and smaller triangulations.

Out of Φ, define for each T_i a φ_i as follows:

- For a vertex x of T_i, pick any $\varphi_i(x) \in \Phi(x)$.

- For a nonvertex x in a trianglet with vertices x_1, x_2, x_3, let $\varphi_i(x)$ be the proper linear comb. of $\varphi_i(x_1)$ $\varphi_i(x_2)$ $\varphi_i(x_3)$.

φ_i is continuous.
So $\exists\, x_i^*$ fixpoint of φ_i.

$$x_i^* = \varphi_i(x_i^*).$$

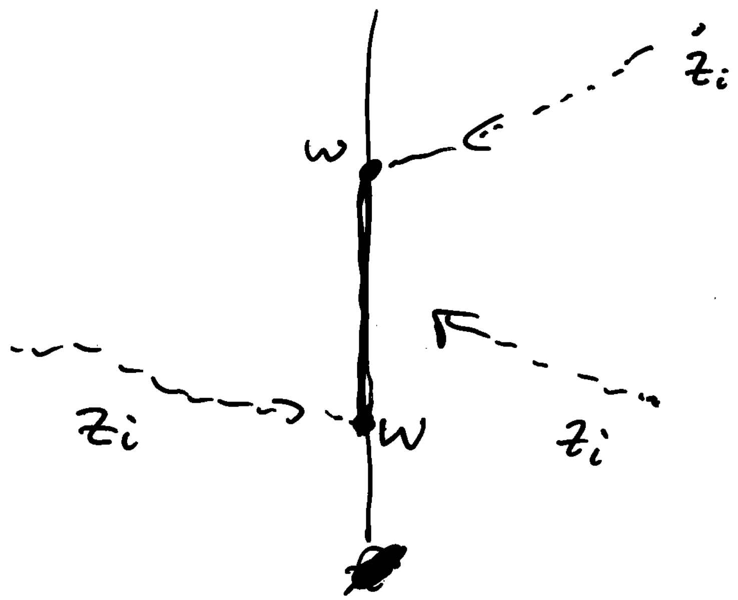
Consider a converging subsequence $\hat{x}_i^*, \rightsquigarrow \hat{x}^*$.

Claim: $\hat{x}^* \in \Phi(\hat{x}^*)$.

$\hat{x}_i^* \longrightarrow x^*$.

Pick a sequence of triangulation vertices corresponding to $\hat{x}_i^*$, call them $\hat{y}_i$.

$\hat{y}_i \longrightarrow x^*$.



Let z_i be the value we chose for T_i at $\hat{y}_i$.

$z_i \in \Phi(\hat{y}_i)$

If w_k is an accumul. point of z_i, then by graph cont, ~~$w \in \Phi(w)$~~ $w_k \in \Phi(x^*)$.

$\hat{x}_i^*$ is in the ~~linear~~ convex comb. of its $\hat{y}_i$.

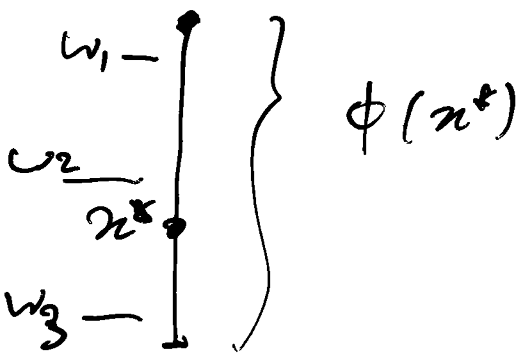
The limit x^* of $\hat{x}_i^*$ must be in the convex combination of the w_k.

So:

$w_k \in \Phi(x^*)$.

x^* is in the convex combination of the w_k.

$\Phi(x^*)$ convex.



So, $x^* \in \Phi(x^*)$.