Secure Equilibria:

$(\pi^*) = (\pi^*_i)$ is secure iff:

$$\forall i \ \forall \pi_i \quad u_i(\pi_i, \pi^*_{-i}) \leq u_i(\pi^*_i, \pi^*_{-i}) \qquad \text{(Nash)}$$

CMH:

$$\forall i, j \ \forall \pi_j . \ u_i(\pi_j, \pi^*_{-j}) \leq u_i(\pi^*) \longrightarrow u_j(\pi_j, \pi^*_{-j}) \leq u_j(\pi^*) .$$

here:

$$\forall i . \ \forall \pi_{-i} : \ u_i(\pi^*_i, \pi_{-i}) \leq \cancel{\pi} u_i(\pi^*) \longrightarrow \exists j . u_j(\pi_{-i}, \pi^*_i) \leq u_j(\pi^*)$$

Evolutionary Equilibrium.

We consider a symmetrical two-player game: for all moves $a, b \in B$, $u_1(a,b) = u_2(b,a)$.

Idea: suppose b^* is a Nash equilibrium.

We want to ~~show~~ show that it is not convenient for players (animals) to deviate from it. Assume a fraction ε of the population deviates and starts playing b.

This is not convenient if:

$$\text{Mutant payoff:} \quad (1-\varepsilon)\, u(b, b^*) + \varepsilon\, u(b, b)$$

$$\leqslant \qquad (1)$$

$$\text{Normal payoff:} \quad (1-\varepsilon)\, u(b^*, b^*) + \varepsilon\, u(b^*, b)$$

Notice that (1) holds ~~fo~~ for all sufficiently small ε iff:

- either $u(b, b^*) \leq u(b^*, b^*)$ (main $1-\varepsilon$ part)

- or $u(b, b^*) = u(b^*, b^*)$,

  and $u(b, b) < u(b^*, b)$.

Therefore, we define:

b^* is an ESS (evolutionarily stable equilibrium) iff:

(b^*, b^*) is a Nash equilibrium

$u(b,b) < u(b^*,b)$ for all $b \in BR(b^*)$, $b \neq b^*$.

Bayesian Games: modeling imperfect information.

- a set N of players $= \{1, \ldots, n\}$.
- a finite set Ω of "states" ($\sim$ game types)

For each $i \in N$:

- a set A_i of actions (moves)
- a finite set T_i of signals, and a function $\tau_i : \Omega \mapsto T_i$.
- a probability measure p_i on Ω (the prior belief of player i) so that $p_i(\tau_i^{-1}(t_i)) > 0$ for all $t_i \in T_i$.
- a preference relation $\succcurlyeq_i$ on the set of probability measures over $A \times \Omega$, where $A = \prod_{i=1}^{n} A_i$.

We define a Nash equilibrium of the ~~strategy ga~~ Bayesian game as the Nash equilibrium of a strategic game G^*, in which the players are

$$(i, t_i) \qquad \text{for all } i \in N,\ t_i \in T_i.$$

"player i, ~~that~~ knows t_i".

The actions of (i, t_i) are still A_i.

So, the set of action tuples is:

$$\times_{i\in N} \left(\times_{t_i \in T_i} A_i \right) .$$

Consider now an action profile

$$a^* = \Bigg(\underbrace{\big((a_1, t_1), (a_2, t_2), (a_3, t_3)\big)}_{\substack{\text{what pl. 1} \\ \text{plays:} \\ a_j, \text{ if signal is } t_j}},\ \underbrace{(\qquad\qquad)}_{\substack{\text{what pl. 2} \\ \text{plays}}},\ \ldots \Bigg)$$

The action profile a^*, together with the signal t_i, induces a probability distribution ~~over~~ $L_i(a^*, t_i)$ over $A \times \Omega$ as follows:

$$L_i(a^*, t_i)\big(\underbrace{(a^*(j, \tau_j(\omega)))_{j \in N}}, \omega\big) = \begin{cases} \dfrac{p_i(\omega)}{p_i(\tau_i^{-1}(t_i))} & \text{if } \omega \in \tau_i^{-1}(t_i) \\ 0 & \text{otherwise.} \end{cases}$$

Move in a^* chosen by player j when receiving the signal $\tau_j(\omega)$. This is the $j, \tau_j(\omega)$ element of a^*.

This is the probability that the state is ω, given that the observation is ~~p_i~~ t_i.

a Nash eq of a Bayesian Game is a Nash eq of the strategic game defined as follows:

G^* players (i, t_i) $\forall i, \forall t_i \in T_i$

Actions of player (i, t_i): A_i.

- The preference relation $\geq_{(i,t_i)}$ is:

$a^* \geq_{(i,t_i)} b^*$ iff

$$L_i(a^*, t_i) \geq_i L_i(b^*, t_i).$$

in Bayes game

Example: second-price auctions.

V = set of values for object.

Bayesian game.

$N = \{1, \dots, n\}$ players.

$\Omega = V^n$ (value of object to each player)

$A_i = \mathbb{R}_{\geq 0}$

$T_i = V$ (every player knows his/her own value only)

$\tau_i(v_1 — v_n) = v_i$.

$p_i(v_1 — v_n) = \prod_{j=1}^{n} q(v_j)$ for some q over V.

(q: prob distr over V)

For a ~~sect~~ ~~ras~~ vector $(v_1 — v_n)$, define:

- winner$(v_1 — v_n)$ = lowest i st. $v_j \leq v_i$ for all $i, j \in N$.
- price $(v_1 — v_n) = \max\limits_{j \in N \setminus \text{winner}(v_1 — v_n)} v_j$.

~~Thm~~

Then, the preference relation of player i can be defined as follows.

Let p_1, p_2 be prob. distr. over $\times_{i \in N} A_i = A$

For $(v_1 \text{—} v_n) \in A$,

$$u_i(v_1 \text{—} v_n) = \begin{cases} 0 & \text{if } i \neq \text{winner}(v_1 \text{—} v_n) \\ v_i - \text{price}(v_1 \cdots v_n) & \text{if } i = \text{winner}(v_1 \text{—} v_n). \end{cases}$$

Then, $p_1 \succcurlyeq_i p_2$

iff $E^{p_1}(u_i) \geqslant E^{p_2}(u_i)$.

<u>Theorem</u>: This game has a Nash eq. where $a^*(i, v_i) = v_i$.

It is a weakly dominant strategy ~~to play~~ for each player to play its own valuation.

This is a truthful equilibrium.

Weekly Dominant Action:

action a_i of player i is weekly dominant if, for all a_{-i},

$$u_i(a_i, a_{-i}) \geq u_i(a_i', a_{-i}).$$

HW: Let $(a_1, \ldots a_n)$ be an action profile consisting of weekly dominant actions. What can you say about $(a_1, \ldots, a_n)$?