A Model of Knowledge

Information function: $P: \ \Omega \mapsto 2^{\Omega}$

If we are in $\omega \in \Omega$, we only know that we are in $P(\omega)$. Of course, usually $\omega \in P(\omega)$. In fact, we assume the following:

P1. $\omega \in P(\omega)$ for all $\omega \in \Omega$.

P2. if $\omega' \in P(\omega)$, then $P(\omega') = P(\omega)$.

P2 is rather strong, and it says that we are considering only P that yield partitions.

An example not satisfying P2:

$\Omega = \mathbb{N} \qquad P(n) = \{ k \mid |n-k| \leq 3 \}$.

Event: $E \subseteq \Omega$

Given Ω, P, we know E at ω if $P(\omega) \subseteq E$.

Knowledge function:

$$K(E) = \{\omega \in \Omega \mid P(\omega) \subseteq E\}$$

$K(E)$: where we know E.

A knowledge function satisfies the following properties:

K1 $\quad K(\Omega) = \Omega$

K2 $\quad$ If $E \subseteq F$, $\quad K(E) \subseteq K(F)$

K3 $\quad K(E) \cap K(F) = K(E \cap F)$.

If P satisfies P1 we also have:

K4 $\quad K(E) \subseteq E$.

If P is partitional (P1 & P2 hold):

K5 $\quad K(E) \subseteq K(K(E))$

K6 $\quad \Omega \setminus K(E) \subseteq K(\Omega \setminus K(E))$

The Puzzle of the Hats.

n people. Each can see only the other people's hat color.

"Hats are black or white." $\underbrace{\text{There is at least a white hat.}}$ I will count slowly. After each number, raise your hand if you know the color of your hat."

Analysis: Ω = set of all $(c_1, \ldots, c_n)$ such that:

$$c_i \in \{W, B\}$$
$$\exists i.\ c_i = W$$

$P_i^1(c)$ = all states that accord with i'th initial observation: if i sees a white hat

$$P_i^1(c) = \{(c_{-i}, W), (c_{-i}, B)\}.$$

two possibilities for i's hat color.

and if i does not see a white hat:

$$P_i^1(c) = \{c\}.$$

In this case, i knows the color of their hat, as: $P_i^1(c) \subseteq \{c' \mid c'_i = W\}$.

So, at round 1, a hand is raised by i only if i does not see a white hat.

Let $F^1 = \{c \mid |\{i \mid c_i = W\}| = 1\}$ the config. where someone raises a hand in the first round. If nobody raises a hand, we know we are in $\Omega \setminus F^1$. There must be at least two white hats.

At the second round, we have:

- If there is i s.t. $c_i = W$, $c_j = W$ for $j \neq i$, and $c_k = B$ for all $k \notin \{i, j\}$, then $P_i^2(c) = (c_j = W, c_i = W, c_{-\{i,j\}})$, and same for j.

- Otherwise,

$$P_i^2(c) = \{(W, c_{-i}), (B, c_{-i})\}.$$

Etc etc, by induction.

Common Knowledge

Ω $\quad K_1$ for person 1 $\quad K_2$ for person 2.

Def A

$E \subseteq \Omega$ is common knowledge in ω if

ω is a member of every set of the type:

$$K_1(E) \qquad K_2(E) \qquad K_1(K_2(E)) \quad K_2(K_1(E))$$

and in general, of every set defined by this grammar:

$$S \longmapsto K_1(E) \mid K_2(E) \mid K_1(S) \mid K_2(S)$$

An alternative definition:

Let P_1 and P_2 be information functions.

- $F \subseteq \Omega$ is <u>self-evident</u> between 1 and 2 if for all $\omega \in F$ we have

$$P_1(\omega) \subseteq F \quad , \quad P_2(\omega) \subseteq F.$$

Def B.

- $E \subseteq \Omega$ is <u>common knowledge</u> betw 1 and 2 in the state $\omega \in \Omega$ if there is a self-evident F for which $\omega \in F \subseteq E$.

Lemma: Consider Ω, P_1, P_2 partitional

P_1 and P_2 induce K_1, K_2.

Let $E \subseteq \Omega$.

The following are equivalent:

1) $K_i(E) = E$ for $i = 1, 2$.

2) E is self-evident betw. 1, 2

3) E is a union of equiv. classes induced by P_i, $i = 1, 2$.

Proof

Assume 1). $\forall\, \omega \in E$, $P_i(\omega) \subseteq E$ for $i = 1, 2$.
This gives 2).

Assume 2). ~~Ea~~ Then $E = \bigcup_{\omega \in E} P_i(\omega)$. This gives 3).

Assume 3). Then 1) follows. (try it).

<u>Theorem</u> Consider Ω, partitional P_1, P_2, and the associated K_1, K_2.

$E \subseteq \Omega$ is common knowledge betw. 1 and 2 at w according to Def A iff it is common knowledge according to Def B.

<u>Proof</u>: Assume E is common knowledge according to Def A.

$$E \supseteq K_1(E) \supseteq K_2(K_1(E)) \supseteq K_1(K_2(K_1(E))) \supseteq \dots$$

As Ω is finite, there must be a fixpoint $F_i \neq \emptyset$ so that $K_j(F_i) = F_i$, and $K_i(F_i) = F_i$ by K4, K5. So, F_i is self-evident, and Def B holds.

Conversely, assume that ω is common knowledge according to Def B. There is a self-evident F st.

$$P_1(F) \subseteq F \quad P_2(F) \subseteq F$$

So by the lemma, $K_1(F) = F$, $K_2(F) = F$. Def A follows easily.

Solving games and knowledge.

<u>We consider two-player games</u> (some results do not hold for more than two players).

$G = \langle (A_i), (u_i) \rangle \qquad i = 1, 2.$

Ω is as follows. Each $\omega \in \Omega$ has, for $i = 1, 2$:

- $P_i(\omega) \subseteq \Omega$
- $a_i(\omega) \subseteq A_i$ (action chosen)
- $\mu_i(\omega)$, a prob. measure on A_{-i}.

Theorem

Assume that in ω, every $i \in \{1,2\}$:

1) knows the other player's moves:

$$P_i(\omega) \subseteq \{\omega' \mid a_{-i}(\omega') = a_{-i}(\omega)\}.$$

2) Has a rational belief:

$$\text{Support}(\mu_i(\omega)) \subseteq \{a_{-i}(\omega') \mid \omega' \in P_i(\omega)\}.$$

3) Is rational: $a_i(\omega)$ is a best response to $\mu_i(\omega)$.

Then, $a(\omega)$ is a Nash equilibrium.

Proof:

By 3), $a_i(\omega)$ is a best response to $\mu_i(\omega)$, and thus, to $a_{-i}(\omega)$ (using 1, 2).

Theorem

Assume that in each $\omega \in \Omega$, every $i \in \{1,2\}$:

1) Knows the other player's belief:

$$P_i(\omega) \subseteq \{\omega' \mid \mu_{-i}(\omega') = \mu_{-i}(\omega)\}$$

2) Has a belief consistent with the knowledge:

$$\mathrm{Supp}\,(\mu_i(\omega)) \subseteq \{a_{-i}(\omega') \mid \omega' \in P_i(\omega)\}$$

3) Knows that the other player is rational: for all $\omega' \in P_i(\omega)$, $a_{-i}(\omega')$ is a best response to ~~...~~ $\mu_{-i}(\omega')$.

Then, the mixed profile $(\mu_2(\omega), \mu_1(\omega))$ is a Nash equilibrium.

Proof:

Let a_i^* be an action of i in the support of ~~...~~ $\alpha_i = \mu_{-i}(\omega)$.

By 2), there is $\omega' \in P_{-i}(\omega)$ where $a_i(\omega') = a_i^*$.

By 3), a_i^* is a best response to $\mu_i(\omega')$, which by 1) is equal to $\mu_i(\omega)$.

The Email Game

	A	B
A	M, M	1, −L
B	−L, 1	0, 0

G_a

$1-p$

	A	B
A	0, 0	1, −L
B	−L, 1	M, M

G_b

p

$p < \frac{1}{2}$.

Only pl. 1 knows whether the game is G_a, G_b.

If it is G_a, they send each other messages and acks, each of which gets lost with prob ε.

They can look at how many messages have been sent, and decide what to play.

~~Let us analyze the case for Gb.~~

$$\Omega = \{ (Q_1, Q_2) \mid Q_1 = Q_2 \text{ or } Q_1 = Q_2 + 1 \}$$

$$\tau_i (Q_1, Q_2) = Q_i \qquad i = 1, 2$$

$$p_i (0,0) = 1 - p$$

$$p_i (q+1, q) = p \varepsilon (1-\varepsilon)^{2q} \qquad q \in \mathbb{N}.$$

$$p_i (q+1, q+1) = p \varepsilon (1-q)^{2q+1}$$

<u>Theorem</u> There is only one Nash equilibrium, (A,A), in the Email game.

<u>Proof</u>: In $(0,0)$, A is dominant for player 1.
So when $T_1(\omega)=0$, (we are in $(0,0)$),
and pl. 1 chooses A.

When $T_2(\omega)=0$, then:

$1-p$ we are in $(0,0)$
or
$p\varepsilon$ we are in $(1,0)$.

If pl. 2 chooses A:

$$\text{payoff} \geq \frac{(1-p)M}{(1-p)+p\varepsilon}$$ whatever pl. 1 chooses in $(1,0)$.

If pl. 2 chooses B: if $(0,0)$ → if $(1,0)$ in the <u>best</u> case.

$$\text{payoff} \leq \frac{-L(1-p)+p\varepsilon M}{(1-p)+p\varepsilon}$$

So, pl. 2 chooses A, which is strictly optimal.

Assume now that for all (Q_1, Q_2) with $Q_1 + Q_2 < 2q$, players 1, 2 choose A.

Consider pl. 1 decision when $T_1(\omega) = q$.
We are in (q, q) or $(q, q-1)$.

$$\Pr(q, q-1) = \frac{\varepsilon}{\varepsilon + (1-\varepsilon)\varepsilon} = z > \frac{1}{2}$$ lost at round $q-1$

$$\Pr(q, q) = \frac{(1-\varepsilon)\varepsilon}{\varepsilon + (1-\varepsilon)\varepsilon} < \frac{1}{2}$$ lost at round q.

General idea: it is more likely that messages are lost sooner

------ ε rather than later — $(1-\varepsilon)\varepsilon$.

Key point!

- B chosen: payoff $\leq z(-L) + (1-z)M < 0$
- A chosen: payoff ≥ 0.

So pl. 1 chooses A when $T_1(\omega) = q$.
The reasoning for pl. 2 is similar. So A is always chosen.