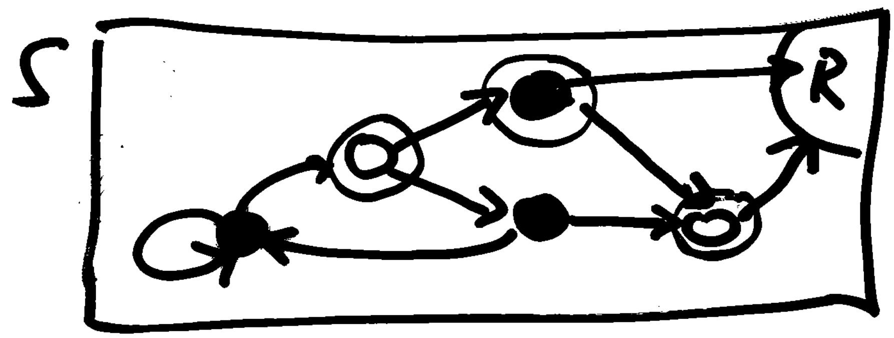
Deterministic Turn-Based Games   0

DTB Games

Reachability:



○ : pl. 1

● : pl. 2

given $X \subseteq S$, let
$Pre_1(X)$ be the set of states from which pl. 1 can force the game to X in 1 move:

$\gamma(s)=1)$ $\exists a \in \Gamma_1(s).\ \tau(s,a) \in X$

$\exists a \in \Gamma_1(s).\ \delta(s,a,-)(X) = 1.$

Defs

$s \in S \qquad a \in \Gamma_i(s)$

$\gamma(s)=1)$ $\tau(s,a) = t$ s.t. $\delta(s,a)(t) = 1.$

$\delta(s,a,-)(t) = 1.$

$\gamma(s)=2)$ $\tau(s,a) = t$ iff

$\delta(s,-,a)(t) = 1.$

$s \xrightarrow{a} t.$

$X \subseteq S$

$s \in Pre_1(X)$ iff:

$\gamma(s)=1$: $\exists a \in \Gamma_1(s). \tau(s,a) \in X$

$\gamma(s)=2$: $\forall a \in \Gamma_2(s). \tau(s,a) \in X$.

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algo for $\langle 1 \rangle \Diamond R$:

$X_0 = \emptyset$

$X_1 = R$

$X_2 = R \cup Pre_1(X_1)$

$X_3 = R \cup Pre_1(X_2)$

$\vdots$ $\quad X_k \subseteq X_{k+1}$. Lemma.

$X^* = \lim_{k \to \infty} X_k$.

f is monotonic if:

$$x \leq y \Rightarrow f(x) \leq f(y).$$

In 2^S, for $A, B \in 2^S$ $(A, B \subseteq S)$,
I say $A \leq B$ iff $A \subseteq B$.

Lemma 1: Pre_1 is monotonic.

$$X \subseteq Y$$
$$\Downarrow$$
$$Pre_1(X) \leq Pre_1(Y)$$

$$X_0 = \emptyset \quad \subseteq \quad R = X_1$$
$$R = R \cup Pre(\emptyset) = X_1 \quad \subseteq \quad R \cup Pre(R) = X_2$$
$$X_2 = R \cup Pre(X_1) \quad \subseteq \quad R \cup Pre(X_2) = X_3$$
$$\dots$$
$$\forall k: \quad X_k \leq X_{k+1}.$$

$X_0 = \emptyset$

$X_1 = R \cup Pre_1(X_0)$

$X_2 = R \cup Pre_1(X_1)$

$\vdots$

$X_* = \lim_{k \to \infty} X_k$.

$X_1 = R$

$X_2 = X_1 \cup Pre_1(X_1)$

$X_3 = X_2 \cup Pre_1(X_2)$

HW1P1

Lemma 2: $X_* \subseteq \langle 1 \rangle \Diamond R$.

Proof: from X_k, $k > 0$,
I can reach R in at most
$k-1$ transitions (by ind). ∎

Winning strategy for X^F:

For $s \in X_{k+1} \setminus X_k$, play
according to $Pre_1(X_k)$.

A_0: $X_0 = \emptyset$

n iter. $\left[\begin{array}{l} \forall k \geq 0:\ X_{k+1} = \underbrace{R \cup Pre(X_k)}_{m} \\ \text{until} \\ \quad X_{k+1} = X_k. \end{array}\right.$

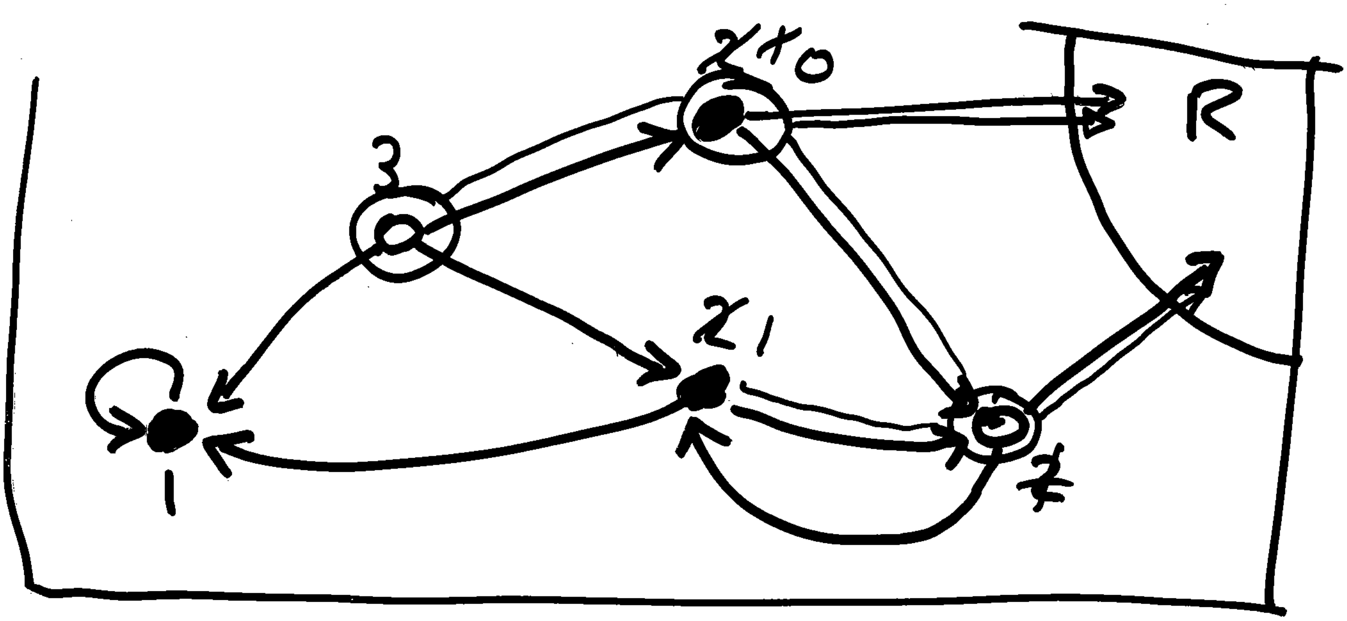
Output: X_k.

Complexity: $O(n \cdot m)$

$|S| = n$

$|G| = m = \sum_{s \in S} |\Gamma_i(s)|$

Better algo:



$O(n) + O(m) = O(m)$

$X_0 = \emptyset$

$X_{k+1} = R \cup Pre_1(X_k) \rightsquigarrow X_*.$

$f : 2^S \mapsto 2^S$

$f(X) = R \cup Pre_1(X).$

$f(X_*) = R \cup Pre_1(X_*) = X_*.$

$f(X_*) = X_*$ $\quad X_*$ is a fixpoint of f.

$f(S) = S$ $\quad S$ " " " ...

X_* is the <u>least</u> fixpoint of f.
(Lemma 3)

Proof of Lemma 3

Let Y be a fp: $f(Y) = Y$.

$$\emptyset \subseteq Y.$$

$$X_1 = f(\emptyset) \subseteq f(Y) = Y \quad f \text{ monot.}$$

$$X_2 = f(X_1) \subseteq f(Y) = Y$$

$$\vdots$$

$$\forall k.\ X_k \subseteq Y.$$

$$\lim_{k\to\infty} X_k \subseteq Y$$

$$X_* \subseteq Y.$$

$f\ P \to Q$. The least fixpoint of f is written

$$\mu x . f(x).$$

μ-CALCULUS (ON SETS)

$\cup$, $\cap$,

$\neg$, Pre_1, Pre_2,

$\mu X . \varphi(X)$

$$\varphi ::= set \mid \varphi_1 \cup \varphi_2 \mid \varphi_1 \cap \varphi_2 \mid \neg \varphi_1 \mid Pre_1(\varphi_1) \mid Pre_2(\varphi_1) \mid \mu X . \varphi(X) \mid \nu X . \varphi(X).$$

$[\varphi] \subseteq S$ is the "meaning" of φ.

$[A] = A$ for $A \subseteq S$.

$$[\varphi_1 \underset{\cap}{\cup} \varphi_2] = [\varphi_1] \underset{\cap}{\cup} [\varphi_2]$$

$$[\neg \varphi_1] = S \setminus [\varphi_1]$$

$$[Pre_1(\varphi)] = Pre_1([\varphi])$$

$[\mu X. \varphi(X)]$ = the least set A such that $[\varphi(A)] = A$.

Compute it as:

$$X_0 = \emptyset$$
$$X_{k+1} = \dot{\varphi}(X_k)$$
$$[\mu X. \varphi(X)] = \lim_{k \to \infty} X_k.$$

$[\nu X. \varphi(X)]$ = the largest A s.t. $[\varphi(A)] = A$.

$$X_0 = S$$
$$X_{k+1} = \varphi(X_k)$$
$$[\nu X. \varphi(X)] = \lim_{k \to \infty} X_k.$$

$\boxed{\text{HW1 P2}}$

Restriction

in $\mu X.\, \varphi(X)$

$\nu X.\, \varphi(X)$

X occurs within an <u>even</u> (0, 2, 4, ...) number of $\neg$.

(So φ is monotonic).

Reachability:

$$\mu X.\,(R \cup Pre_1(X)) \subseteq \langle 1 \rangle \Diamond R.$$

~ will show:

$$\neg \mu X.\,(R \cup Pre_1(X)) \subseteq \langle 2 \rangle \Box \neg R.$$