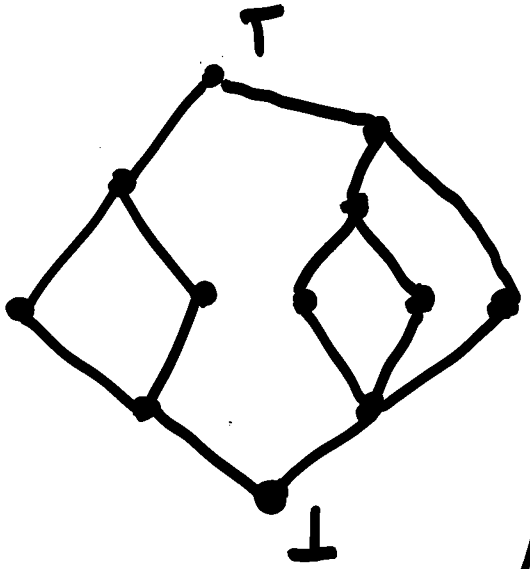
vv

# LATTICE






## PARTIAL ORDER

$(L, \leqslant)$

$\leqslant$: Reflexive

$\forall a : a \leq a$

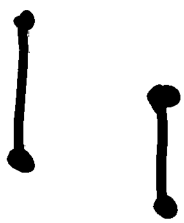
: Transitive

$a \leq b \wedge b \leq c$

$\rightarrow a \leq c$

Antisymmetric:

$a \leq b \wedge b \leq a \rightarrow a = b.$



LATTICE:

A Partial order $(L, \leq)$ is a <u>lattice</u> if, $\forall a, b \in L$,

$a \sqcup b$ and $a \sqcap b$ exist.

$\uparrow$ least upper bound

$\uparrow$ greatest lower bound.

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$a$ is an upper bound of $B \subseteq L$ iff, $\forall b \in B$, $a \geq b$.

$a$ is a l.u.b. of $B$ if $a$ is an ub of $B$, and if for all other ub $a'$, $a \leq a'$.

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A <u>complete</u> lattice is a $(L, \leq)$ where, for all $B \subseteq L$, $\sqcup B$, $\sqcap B$ exist.

# KNASTER - TARSKI THM.

a monotonic function f over a (*) lattice has a greatest & least fixed points.

(*) complete.

<u>Def</u> for monotonic $f$,

$f$ over a complete lattice $(L, \leq)$ is

<u>continuous</u> if, $\forall B \subseteq L$,

$$f(\sqcup B) = \sqcup \{ f(b) \mid b \in B \}.$$

(This is "left continuity", for "right", use $\sqcap$).

Thm:

- Lattice $(L, \leq)$ complete.
- $f$ monotonic, continuous.
  $f : L \to L$.

Then,

$$\mu x . f(x) = \lim_{k \to \infty} f^k(\bot).$$

Proof

let $x_k = f^k(\bot)$.

let $x_* = \sqcup \{x_k\}$.

1) $x_* \leq \mu x . f(x)$. $\boxed{HW}$

2) $f(x_*) = x_*$. In fact:

$f(x_*) = f(\sqcup \{x_k\}) =$ by continuity

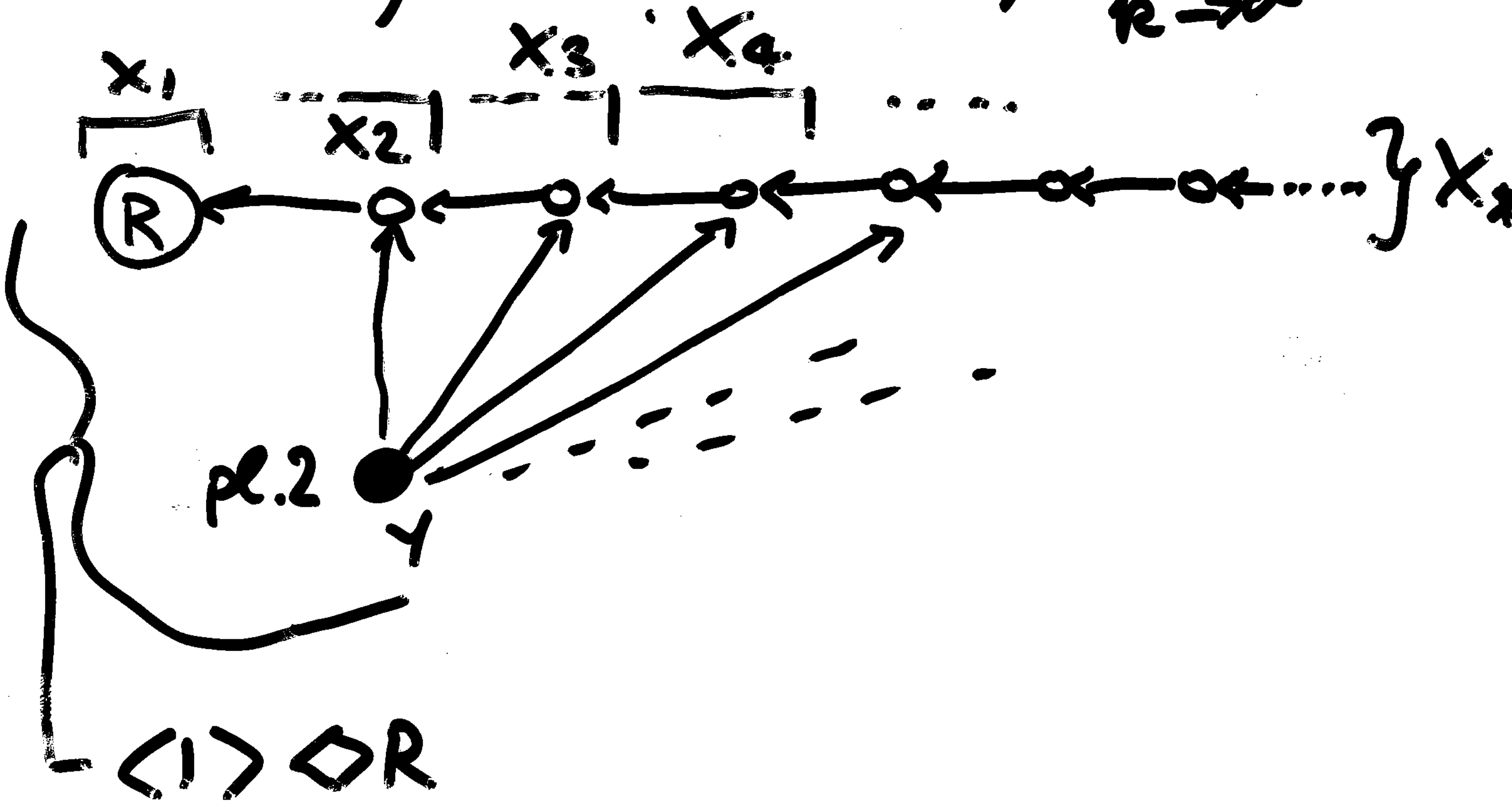
$$= \sqcup \{ f(x_k) \mid k \geq 0 \}$$
$$= \sqcup \{ f^{k+1}(\bot) \mid k \geq 0 \}$$
$$= \sqcup \{ f^k(\bot) \mid k > 0 \} = x_* .$$

An exemple :

Pre is not continuous if $S$ is infinite.

In the exemple,

- $\mu X.(R \cup Pre(X))$ is $= \langle 1 \rangle \Diamond R$
- but $\mu X.(R \cup Pre(X)) \neq \lim_{k \to \infty} X_k$.




$$\mu X.(R \cup Pre(X)) = X_* \cup Y$$

Complement in a complete lattice of subsets:

$$\neg(A \cup B) = \neg A \cap \neg B$$

$\neg A = T \setminus A$ $\qquad$ $T$ : top.

$$\neg(A \cup B) = \neg A \cap \neg B$$
$$\neg(A \cap B) = \neg A \cup \neg B$$

$f(X)$, where $X$ has "positive polarity" (even number of enclosing $\neg$).

$\mu X.\, f(X)$.

$$\neg \mu X.\, f(X) = \nu X.\, \neg f(\neg X)$$
$$\neg \nu X.\, f(X) = \mu X.\, \neg f(\neg X)$$

$\boxed{\text{HW}}$

$$\mu X.\,(R \cup Pre_1(X)) \quad \subseteq \langle 1 \rangle \Diamond R.$$

$$\neg \mu X.\,(R \cup Pre_1(X)) =$$

$$= \nu X.\,\neg(R \cup Pre_1(\neg X))$$

$$= \nu X.\,(\neg R \cap \neg Pre_1(\neg X))$$

$\gamma(s) = 1$:

$s \in Pre_1(X)$ iff $\exists a \in \Gamma_1(s).\ \tau(s,a) \in X.$

$s \in \neg Pre_1(\neg X)$ iff

$\neg \exists a \in \Gamma_1(s)\ \tau(s,a) \in \neg X$

$\forall a \in \Gamma_1(s).\ \tau(s,a) \notin \neg X$

$\forall a \in \Gamma_1(s).\ \tau(s,a) \in X.$

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$\gamma(s) = 2$:

$s \in \neg Pre_1(\neg X)$ iff

$\exists a \in \Gamma_2(s).\ \tau(s,a) \in X.$

$\neg \cancel{\exists a(\Gamma_1(s}$

$\neg \exists a(a \in \Gamma_1(s) \wedge \tau(s,a) \in \neg X)$

$\forall a(a \notin \Gamma_1(s) \vee \tau(s,a) \notin \neg X)$

$\forall a(a \notin \Gamma_1(s) \vee \tau(s,a) \in X)$

$\forall a(a \in \Gamma_1(s) \rightarrow \tau(s,a) \in X)$

$\forall a \in \Gamma_1(s).\ \tau(s,a) \in X.$

So, $\neg Pre_1(\neg X) = Pre_2(X)$

$\neg Pre_2(\neg X) = Pre_1(X)$

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$$\neg \mu X. (R \cup Pre_1(X)) =$$

$$= \nu X. (\neg R \cap Pre_2(X)).$$

$X_0 = S$

$X_1 = \cancel{S \cap} \neg R \cap Pre_2(S) = \neg R$

$X_2 = \neg R \cap Pre_2(X_1)$

$X_3 = \neg R \cap Pre_2(X_2)$

If $s \in X_k$, pl. 2 can force to be in $\neg R$ for at least $k-1$ steps.

$$X_* = \lim_{k \to \infty} X_k .$$

Pl. 2 from $X_*$ can be in $\neg R$ forever.

FOR D, TB GAMES

<u>**THM**</u>

$$\langle 1 \rangle \Diamond R \overset{?}{=} \mu X. (R \cup Pre_1(X))$$

$$\neg$$

$$\langle 2 \rangle \Box \neg R \overset{?}{=} \nu X. (\neg R \cap Pre_2(X))$$

Proof:

<u>Corollary</u> (game $\Diamond, \Box$ are determined)

$$\langle 1 \rangle \Diamond R = \neg \langle 2 \rangle \Box \neg R$$

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