$$\langle 1 \rangle \Diamond R = \mu X.\ (R \cup \mathrm{Pre}_1(X)) \qquad (1)$$

<u>Proof</u>

$$X_0 = \emptyset$$
$$X_1 = R \cup \mathrm{Pre}_1(\emptyset)$$
$$X_2 = \quad \vdots$$

By induct: X_k can reach in $\leq k$ steps.

$$\langle 2 \rangle \Box \neg R = \nu X.\ (\neg R \cap \mathrm{Pre}_2(X)) \qquad (2)$$

Let W be any fixpoint of (2).

$$W = \neg R \cap \mathrm{Pre}_2(W).$$

hence

$$\begin{tikzcd}
W \arrow[r, "\text{step}"] \arrow[d, Rightarrow] & W \arrow[r] \arrow[d, Rightarrow] & W \arrow[r] \arrow[d, Rightarrow] & \cdots \\
\neg R & \neg R & \neg R & \cdots
\end{tikzcd}$$

So, $W \subseteq \langle 2 \rangle \Box \neg R$.

<u>Hint</u> Proving properties of fixpoints:

μ : induction

ν : assume you have <u>a</u> fixpoint, and reason on it.

Different flavor of μ, ν:

$$X = R \cup Pre(X)$$

$$X \xrightarrow{\vee R} X \xrightarrow{\vee R} X \xrightarrow{\vee R} X \cdots \cdots \overset{\mu}{\xrightarrow{R}} X^{\mu_0}$$

$$X = R \cap Pre(X)$$

$$X \xrightarrow{\wedge R} X \xrightarrow{\wedge R} X \cdots\cdots$$

Büchi games.

$B \subseteq S$ $\quad\quad$ $\langle 1 \rangle \Box \Diamond B$

forever, we reach a B.

= there are ∞ many passages in B, or

B is visited ∞ often.

Generalized Büchi

$B_1 \ldots B_n \subseteq S$ $\quad\quad$ $\bigwedge_{i=1}^{n} \Box \Diamond B_i$.

Gen Büchi $\quad\quad\quad\quad$ Büchi

$\langle S, \ldots \rangle$ $\quad \Longrightarrow \quad$ $S' = S \times \underset{k}{\{1 \ldots n\}} \times \underset{d}{\{0,1\}}$

$B_1 \ldots B_n$

$\tau(s,a) = t$ $\quad\quad$ $\tau((s,k,d),a) =$

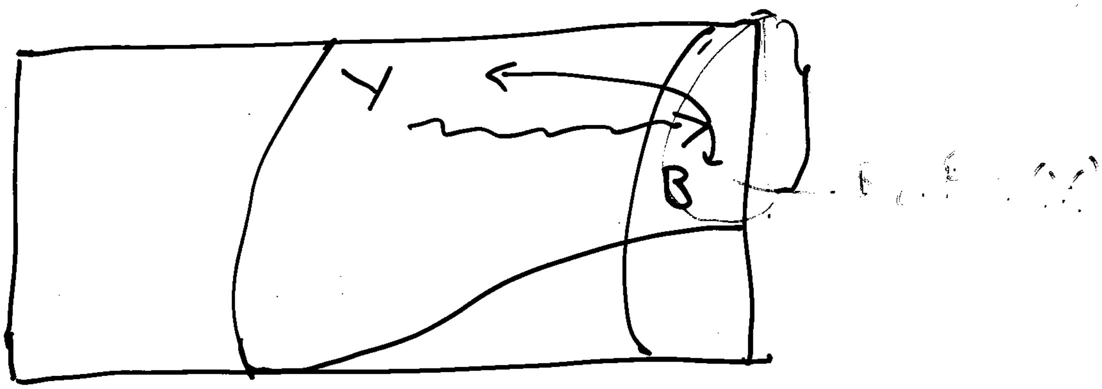
$=$
- if $t \notin B_k$, $\quad (t, k, 0)$
- if $t \in B_k$ $\quad (t, k', d')$

$k' = (k \bmod n) + 1$

$d' = 1$ if $k' = 1$

$\quad 0$ otherwise.

$\langle 1 \rangle\ \Box\Diamond B$?



assume we guess Y such that:

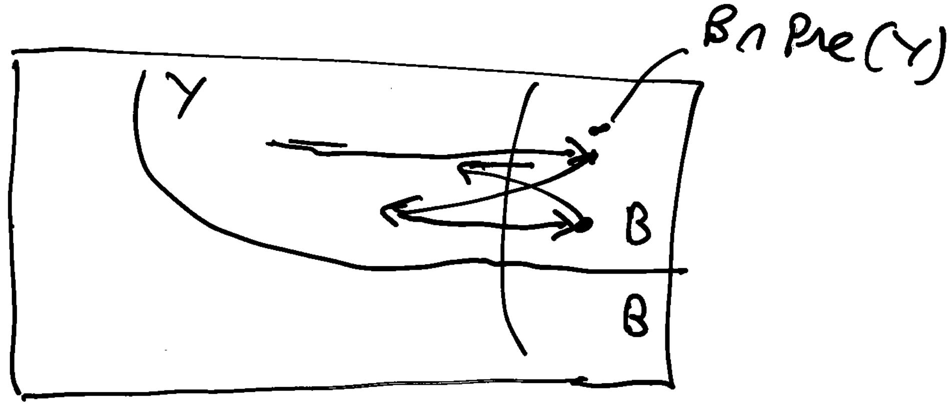
1) $Y \subseteq \langle 1 \rangle \Diamond (B \wedge Y)$

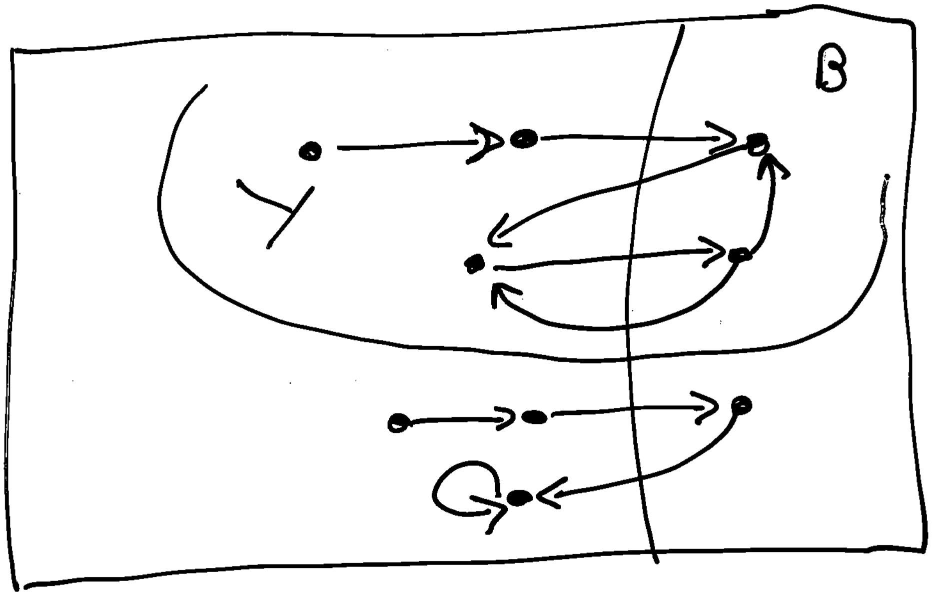
2) $B \wedge Y \leadsto Y$

Then, $Y \subseteq \langle 1 \rangle \Box\Diamond B$.

1) and 2) can be written as:

$$Y = \mu X. \left(\begin{array}{c} \mathrm{Pre}_1(X) \\ \cup \\ B \cap \mathrm{Pre}_1(Y) \end{array} \right) \begin{array}{l} \} \text{ getting there.} \\ \\ \} \text{ where to go} \end{array}$$





So, if we find

$$Y = \mu X. \left(\begin{array}{c} Pre_1(X) \\ \cup \\ B \cap Pre_1(Y) \end{array} \right) = f(Y)$$

then $Y \subseteq \langle 1 \rangle \Box \Diamond B$.

Which, of these fixpoints, we like best? $\downarrow$ the greatest! $\downarrow$ $\nu Y. f(Y)$

$$f(Y) = \mu X. \left(\begin{array}{c} Pre_1(X) \\ \cup \\ B \cap Pre(Y) \end{array} \right)$$

So, to compute f, y need to compute a fixpoint.

$$\nu Y. \mu X. \begin{pmatrix} Pre_1(X) \\ \cup \\ B \cap Pre_1(Y) \end{pmatrix}$$

$Y = f(Y)$ $\qquad$ $f = \mu X. \begin{pmatrix} Pre_1(X) \\ \cup \\ B \cap Pre_1(Y) \end{pmatrix}$

$\nu Y. f(Y).$

$Y_0 = S$ $\qquad$ $f(Y_0)$ $\qquad$ $X_0 = \emptyset$

$$X_1 = \begin{pmatrix} \overbrace{Pre_1(X_0)}^{\emptyset} \\ \cup \\ B \cap \underbrace{Pre_1(S)}_{S} \end{pmatrix} = B$$

$\vdots$

$Y_1 = \langle 1 \rangle \Diamond B. \xleftarrow{\quad} X_* = \langle 1 \rangle \Diamond B.$

$X_0 = \emptyset$

$$X_1 = \begin{pmatrix} Pre_1(X_0) \\ \cup \\ B \cap Pre_1(Y_1) \end{pmatrix}$$

$$= \begin{pmatrix} Pre_1(X_0) \\ \cup \\ B \cap Pre_1(\langle 1 \rangle \Diamond B) \end{pmatrix}$$

$\vdots$

$Y_2 \xleftarrow{\quad} X_*$

$\vdots$

$$\nu Y.\, \mu X. \left(\begin{array}{c} \cancel{B}\, Pre_1(X) \\ \cup \\ B \cap Pre_1(Y) \end{array} \right)$$

$Y_0 = S$

repeat {

$\quad X_0 = \phi$

$\quad$ repeat {

$$X_{k+1} = \left(\begin{array}{c} Pre_1(X_k) \\ \cup \\ B \cap Pre_1(Y_j) \end{array} \right)$$

$$\nu Y.\, \mu X. \left(\begin{array}{c} Pre_1(X) \\ \cup \\ B \cap Pre_1(Y) \end{array} \right)$$

$Y' = S$
repeat {
    $Y := Y'$
    $X' := \emptyset$
    repeat {
        $X := X'$
        $X' = \left(\begin{array}{c} Pre_1(X) \\ \cup \\ B \cap Pre_1(Y) \end{array} \right)$
    } until $X = X'$
    $Y' = X$
} until $Y' = Y$
return Y.

Complexity:

Dumb: $O(n^2 \cdot m)$

Smart: $O(n \cdot m)$

n iter of Y.

reach $B \cap Pre_1(Y)$

$$\nu Y.\, \mu X. \begin{pmatrix} \neg B \cap \mathrm{Pre}_1(X) \\ \cup \\ B \cap \mathrm{Pre}_1(Y) \end{pmatrix} \qquad (B)$$

[While we compute,

$X \subseteq Y.$

$\mathrm{Pre}(X) \subseteq \mathrm{Pre}(Y).$

In $\neg B$, we must make true $\mathrm{Pre}(X)$

B we can just make true $\mathrm{Pre}(Y)$.]

[another explanation:

to reach R,

$$\mu X. \begin{pmatrix} \neg R \cap \mathrm{Pre}(X) \\ \cup \\ R \end{pmatrix}$$

]

$$\neg \nu Y. \mu X. \begin{pmatrix} \neg B \cap Pre_1(X) \\ \cup \\ B \cap Pre_1(Y) \end{pmatrix}$$

$$\mu Y. \nu X. \begin{pmatrix} \neg B \cap Pre_2(X) \\ \cup \\ B \cap Pre_2(Y) \end{pmatrix} \quad (coB)$$

Because:

$$\mu X. \nu Y. \begin{pmatrix} \neg B \cap \neg Pre_1(\neg X) \\ \cup \\ B \cap \neg Pre_1(\neg Y) \end{pmatrix}$$

We must show that:

$$(coB) \subseteq \langle 2 \rangle \Diamond \Box \neg B.$$

nifty trick:

$$\Phi = \begin{pmatrix} B_1 \cap \varphi_1 \\ \cup \\ B_2 \cap \varphi_2 \\ \cup \\ \vdots \\ B_n \cap \varphi_n \end{pmatrix} = \begin{pmatrix} B_1 \to \varphi_1 \\ \cap \\ B_2 \to \varphi_2 \\ \cap \\ \vdots \\ B_n \to \varphi_n \end{pmatrix}$$

$B_1, B_2, \ldots B_n$ $\quad$ $\bigcup B_i = S$ $\quad$ partitions

$B_j \cap B_i = \phi$

$$\neg \Phi = \begin{pmatrix} B_1 \to \neg \varphi_1 \\ \wedge \\ B_2 \to \neg \varphi_2 \\ \wedge \\ \vdots \\ B_n \wedge \neg \varphi_n \end{pmatrix} = \begin{pmatrix} B_1 \cap \neg \varphi_1 \\ \cup \\ B_2 \cap \neg \varphi_2 \\ \vdots \\ B_n \cap \neg \varphi_n \end{pmatrix}$$