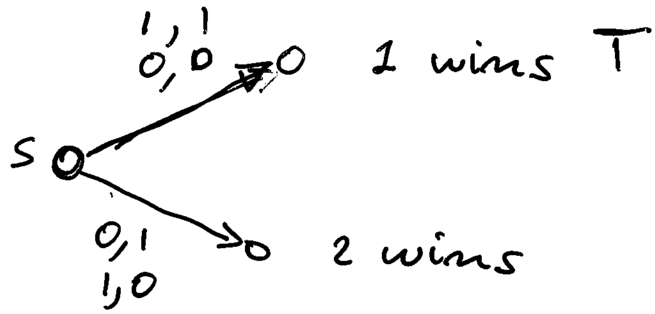


no, too good for pl. 2

$$\exists a \in \Gamma_1(s)\ \forall b \in \Gamma_2(s).\ \tau(s,a,b) \in T.$$

$$\forall b \in \Gamma_2 - .\ \exists a \in \Gamma_1\ .\ \tau(s,a,b) \in T$$

no, too good for pl. 1

$$\underbrace{\inf_x \sup_y f(x,y)}_{g(x)} \geq \sup_y \inf_x f(x,y)$$

$g(x)$ tells me, for each x, how ~~low~~ high y can be on f.

Need

$$\begin{pmatrix} \exists a \\ \forall b \end{pmatrix} .\ \tau(s,a,b) \in T.$$

??

$\forall x\,\exists y \qquad y = f(x)$

$\uparrow$ Skolem functions, —— φ

$\forall u\,\exists w \qquad w = g(u)$

$\forall x\,\exists y\,\forall u\,\exists w \cdot \varphi$

$y = f(x)$

$w = g(u, x)$

$\uparrow$ no

Henkin Quantifiers

$$\begin{pmatrix}\forall x\,\exists y\\ \forall u\,\exists w\end{pmatrix}\cdot\varphi$$

Mixed Move

x at s is $\quad x \in \mathrm{Distr}(\Gamma_1(s))$.

$D_1(s) = \mathrm{Distr}(\Gamma_1(s))$
= set of pl-1 mixed moves at s.

With pure moves:

$$\mathrm{Pre}_1(R) = \{ s \mid \exists a \in \Gamma_1(s).\ \forall b \in \Gamma_2(s).\ \tau(s,a,b) \in R \}$$

We need:

pure $\downarrow$ mixed

sets $\subseteq 2^S$ $\downarrow$ valuations $\underbrace{S \mapsto [0,1]}_{\mathcal{V}}$.

Consider $v \in \mathcal{V}$.

$\mathrm{QPre}: \mathcal{V} \mapsto \mathcal{V}$.

$$\mathrm{QPre}_1(v)(s) =$$

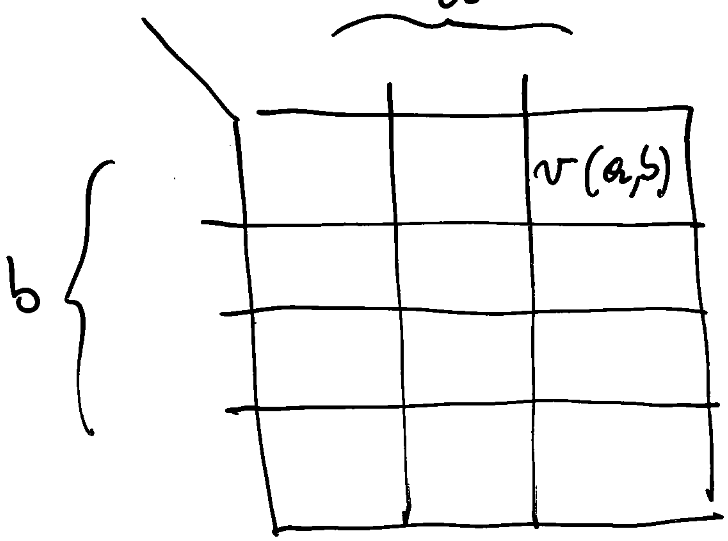
$$= \sup_{x \in D_1(s)} \inf_{y \in D_2(s)} E_s^{x,y}(v)$$

$$E_s^{x,y}(v) = \sum_{t \in S} \sum_{a \in \Gamma_1(s)} \sum_{b \in \Gamma_2(s)} v(t) \cdot \tau(s,a,b)(t) \cdot x(a) \cdot y(b)$$

$$\mathrm{QPre}_1(v)(s) = \sup_{\substack{X \\ \in D_1(s)}} \inf_{\substack{Y \\ \in D_2(s)}} \sum_{a,b} \underbrace{v(s,a,b)}_{\sum_t \tau(s,a,b)(t) \cdot v(t)} \cdot X(a) \cdot Y(b)$$

(zerosum)

Matrix Game (fix s)



von Neumann ('29):

$$\sup_X \inf_Y \underbrace{E^{X,Y}(v)}_{\text{matrix game}} = \underbrace{\inf_Y \sup_X E^{X,Y}(v)}_{\text{often called } \mathrm{val}(v)}.$$

$R \subseteq S$. <u>Reachability</u> Lattice: $(2^S, \subseteq)$

Turn-Based

$$\mu X. \left(R \cup Pre(X) \right)$$

$X_0 = \emptyset$

$X_1 = R$

$X_2 = R \cup Pre_1(R)$

...

Quantitative (simultaneous) $(S \mapsto [0,1], \text{pointwise})$

$$\mu X. \left([R] \sqcup QPre(X) \right)$$

$f \geq g$ iff $\forall s. f(s) \geq g(s)$

$$[R](s) = \begin{cases} 1 & \text{if } s \in R \\ 0 & s \notin R. \end{cases}$$

$p \sqcap q = \cancel{\max} (p, q)$

$p \sqcup q = \max (p, q)$

$[A] \sqcup [B] = [A \cup B]$

$X_0 = \bar{0}$ $(\lambda s.0)$

$X_1 = [R]$

$X_2 = [R] \sqcup QPre_1([R])$

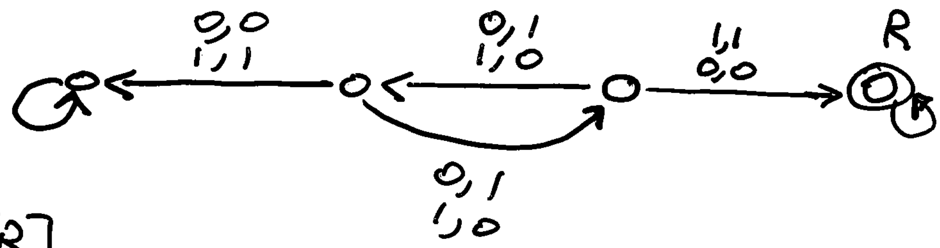
$X_3 = [R] \sqcup QPre(X_2)$

...

$X_{i+1}(s)$: prob of reaching R in at most i steps from s.

$$\lim_{i \to \infty} X_i = X_*$$

$X_*(s) = $ max prob of $\Diamond R$.



$X_1 = [R]$

$X_1 =$	0	0	0	1
$X_2 = [R] \sqcup \mathrm{qPre}(X_1)$				
	0	0	$1/2$	1
$X_3 = [R] \sqcup \mathrm{qPre}(X_2)$				
	0	$1/4$	$1/2$	1
$X_4 =$	0	$1/4$	$\frac{5}{8}$	1
$X_5 =$	0	$\frac{5}{16}$	$\frac{5}{8}$	1
$X_6 =$	0	$\frac{5}{16}$	$\frac{21}{32}$	1
		$\vdots$	$\vdots$	

Thm: $\lim_{i \to \infty} X_i = X_*$, with $X_* \leq \underbrace{\langle\langle 1 \rangle\rangle \Diamond R}$
Prob of pl. 1 reaching R.

$$\langle\langle 1 \rangle\rangle \Diamond R = \sup_{\pi_1} \inf_{\pi_2} Pr^{\pi_1, \pi_2}(\Diamond R).$$

Proof

I show that, $\forall \varepsilon > 0$, $\exists \pi_1$ s.t.

$$\forall s.\ \forall \pi_2.\ Pr_s^{\pi_1 \pi_2}(\Diamond R) \geq X_* - \varepsilon$$

Take ε, and choose i st.

$$X_i(s) \geq X_*(s) - \varepsilon.$$

Call $x^{(i)}$ the mixed move (mapping from states to) that was optimal for X_i, that is:

$$X_i = \sup_{x} \inf_{y} E^{x,y}(X_{i-1}) \quad \sqcup [R]$$

$x^{(i)}$ realizes this value. (max min)

Take π_1 as: $x^{(i)}, x^{(i-1)}, x^{(i-2)} \ldots x^{(1)}, \underbrace{x^{(1)}}_{\text{any.}}$

Take i s.t. $X_i(s) \geq X_*(s) - \varepsilon$.

Let $\xi_i : S \longmapsto \text{Distr}(\text{Moves})$

"selector"

be such that, for all t,

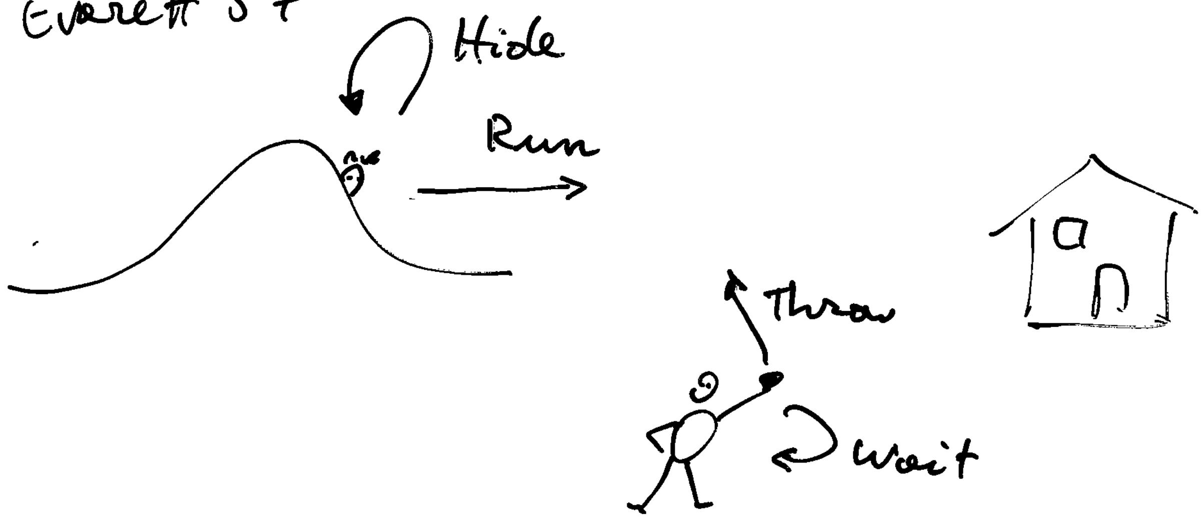
$\xi_i(t)$ realizes the optimal value in

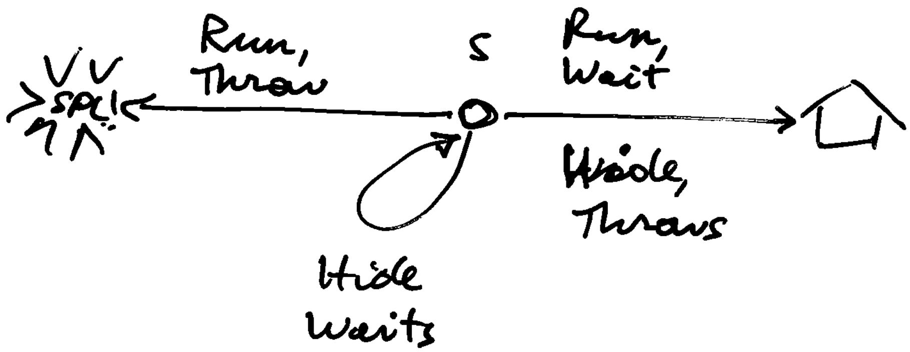
$$X_i(t) = \sup_{\substack{x \in D_1(t) \\ \uparrow \\ \xi_i(t).}} \inf_{Y \in D_2(t)} E^{x,Y}(X_{i-1})$$

ξ_i is a memoryless strategy, in some sense.

$$\pi_1 = (\underset{\text{now}}{\xi_i},\ \underset{\text{1 round}}{\xi_{i-1}},\ \overset{\text{2 rounds}}{\xi_{i-2}},\ \overset{\text{3 rounds}}{\xi_{i-3}},\ \ldots,\ \xi_1,\ \ldots)$$

at round k of the game, play ~~ξ~~ $\xi_{\max(0,i-k)}$.

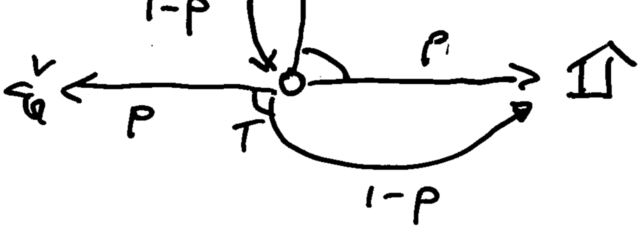




$\langle\langle 1 \rangle\rangle \Diamond \mathrm{Home}\ (s) = 1$? NO:

$$\forall \pi_1 \exists \pi_2.\ \Pr_s^{\pi_1 \pi_2}(\Diamond R) < 1.$$

$\pi_1(p)$: run with prob. p at each round.



$\pi_1(p)$ wins with prob $1-p$ for all $p>0$.

So, $\sup_{\pi_1} \inf_{\pi_2} \Pr_s^{\pi_1 \pi_2}(\Diamond \mathrm{Home}) = 1 = \sup_{p>0} 1-p.$